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PERTURBATION OF BANACH SPACE OPERATORS WITH A COMPLEMENTED RANGE

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Abstract. Let $\mathcal{C}[\mathcal{X}]$ be any class of operators on a Banach space $\mathcal{X}$, and let $Holo^{-1}(\mathcal{C})$ denote the class of operators A for which there exists a holomorphic function f on a neighbourhood $\mathcal{N}$ of the spectrum $\sigma(A)$ of A such that f is non-constant on connected components of $\mathcal{N}$ and $f(A)$ lies in $\mathcal{C}$. Let $\mathcal{R}[\mathcal{X}]$ denote the class of Riesz operators in $\mathcal{B}[\mathcal{X}]$. This paper considers perturbation of operators $A \in \Phi_+(\mathcal{X}) \cup \Phi_-(\mathcal{X})$ (the class of all upper or lower [semi] Fredholm operators) by commuting operators in $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$. We prove (amongst other results) that if $\pi_B(B) = \prod_{i=1}^m (B - \mu_i)$ is Riesz, then there exist decompositions $\mathcal{X} = \oplus_{i=1}^m \mathcal{X}_i$ and $B = \oplus_{i=1}^m B|_{\mathcal{X}_i} = \oplus_{i=1}^m B_i$ such that: (i) If $\lambda \neq 0$, then $\pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda\mu_i)^{\alpha_i} \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$) if and only if $A - \lambda B_0 - \lambda\mu_i \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$), and (ii) (case $\lambda = 0$) $A \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$) if and only if $A - B_0 \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$), where $B_0 = \oplus_{i=1}^m (B_i - \mu_i)$; (iii) if $\pi_B(A, \lambda) \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$) for some $\lambda \neq 0$, then $A - \lambda B \in \Phi_+(\mathcal{X})$ (resp., $\in \Phi_-(\mathcal{X})$).

0. Introduction

Given an infinite-dimensional complex Banach space $\mathcal{X}$, let $\mathcal{B}[\mathcal{X}]$ denote the algebra of operators (equivalently, bounded linear transformations) of $\mathcal{X}$ into itself. Let $A^{-1}(0)$ and $A(\mathcal{X})$ denote, respectively, the null space and the range of an operator $A \in \mathcal{B}[\mathcal{X}]$. The operator A has an *inner generalized inverse* if there exists an operator $B \in \mathcal{B}[\mathcal{X}]$ such that $ABA = A$. It is easily seen that if B is an inner generalized inverse of A, then AB is a projection from $\mathcal{X}$ onto $A(\mathcal{X})$ and $I_{\mathcal{X}} - BA$ is a projection from $\mathcal{X}$ onto $A^{-1}(0)$: Indeed, A is *inner regular* (i.e., A has an inner generalized inverse) if and only if $A(\mathcal{X})$ and $A^{-1}(0)$ are complemented (in $\mathcal{X}$). The study of inner regular operators has a long and rich history, and there is a large body of information available on inner regular operators in the extant literature(see, for example, [7]). An important class of inner regular Banach space operators is that of operators $A \in \mathcal{B}[\mathcal{X}]$ which are either *left or right Fredholm.* Here we say that $A \in \mathcal{B}[\mathcal{X}]$ is left Fredholm, $A \in \Phi_\ell(\mathcal{X})$ (resp, right Fredholm, $A \in \Phi_r(\mathcal{X})$), if $A \in \Phi_+(\mathcal{X})$ and $\mathcal{R}(A)$ is complemented (resp., $A \in \Phi_-(\mathcal{X})$ and $A^{-1}(0)$ is complemented), $\Phi_+(\mathcal{X}) = \{A \in \mathcal{B}[\mathcal{X}] : A(\mathcal{X}) \text{ is closed and } \dim(\mathrm{A}^{-1}(0)) < \infty\}$ is the class of upper semi-Fredholm operators and $\Phi_-(\mathcal{X}) = \{A \in \mathcal{B}[\mathcal{X}] : \dim(\mathcal{X}/\mathrm{A}(\mathcal{X})) < \infty\}$ is the class of lower semi-Fredholm operators (see, e.g.; [12]).

The problem of the perturbation of inner regular operators by compact operators is of some interest, and has been considered in the not too distant past. Thus if an $A \in \mathcal{B}[\mathcal{X}]$ is left Fredholm (or right Fredholm), and $S \in \mathcal{B}[\mathcal{X}]$ is a compact operator, then $A + S$ is left Fredholm (resp., right Fredholm) [10, 5]. This result is in a way the best possible: If $A \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ for Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, $A^{-1}(0)$ is infinite dimensional and complemented in $\mathcal{X}$, $A(\mathcal{X})$ is closed, complemented and of infinite codimension in $\mathcal{Y}$, then the closure of $(A + S)(\mathcal{X})$ is complemented in $\mathcal{Y}$ for every

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compact $S \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ only if $A(\mathcal{X})$ has a complementary subspace isomorphic to a Hilbert space [10, Theorem 3].

For an operator $A \in \mathcal{B}[\mathcal{X}]$, let $\mathcal{H}(\sigma(A))$ denote the set of functions f which are holomorphic on a neighbourhood $\mathcal{N}$ of the spectrum $\sigma(A)$ of A, and let $\mathcal{H}_c(\sigma(A) = \{f \in \mathcal{H}(\sigma(A)) : f \text{ is non-constant on the connected components of } \mathcal{N}\}$. Let $\mathcal{K}[\mathcal{X}]$ denote the ideal of compact operators, and let $\mathcal{R}[\mathcal{X}]$ denote the class of Riesz operators (i.e., operators whose nonzero translates are Fredholm). The operator A is holomorphically compact (resp., Riesz), $A \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$ (resp., $A \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$), if there exists an $f \in \mathcal{H}_c(\sigma(A))$ such that $f(A)$ is compact (resp., Riesz).

This paper considers perturbation of operators in $\Phi_{\pm}(\mathcal{X}) = \Phi_+(\mathcal{X}) \cup \Phi_-(\mathcal{X})$ by commuting operators in $(Holo^{-1}(\mathcal{K}[\mathcal{X}])$, more generally, $Holo^{-1}(\mathcal{R}[\mathcal{X}])$. It is known that if $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$ (resp., $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$), then there exists a polynomial $\pi_B(z) = \prod_{i=1}^m (z-\mu_i)^{\alpha_i}$ for some complex numbers μ_i and positive integers α_i (resp., $\pi_B(z) = \prod_{i=1}^m (z_i-\mu_i)$), which is the minimal polynomial $\pi_B(.)$ of B, such that $\pi_B(B)$ is compact (resp., Riesz).

Let $\Phi_\times(\mathcal{X})$ denote either of $\Phi_+(\mathcal{X})$ and $\Phi_-(\mathcal{X})$. We prove (a more general version of the result) that if $\pi_B(A) \in \Phi_\times(\mathcal{X})$, if $[A,B] = AB - BA = 0$ (or, more generally, $[A,B]$ is in the "perturbation class" $\mathrm{Ptrb}(\Phi_\times(\mathcal{X}))$ of $\Phi_\times(\mathcal{X})$) and $\pi_B(B)$ is Riesz, then $A - B \in \Phi_\times(\mathcal{X})$. The hypothesis $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$ (or, $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$) enforces a decomposition $\mathcal{X} = \bigoplus_{i=1}^m \mathcal{X}_i$ of $\mathcal{X}$ such that $B = \bigoplus_{i=1}^m B_i = \bigoplus_{i=1}^m B|_{\mathcal{X}_i}$ with $\bigoplus_{i=1}^m (B_i-\mu_i)^{\alpha_i}$ compact (resp., $\bigoplus_{i=1}^m (B_i-\mu_i)$ Riesz). Let $B_0 = \oplus_{i=1}^m (B_i-\mu_i)$, where m and μ_i are as above. It is proved that if $[A,B]=0$ and $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$, then: (a) $\pi_B(A,\lambda) = \prod_{i=1}^m (A-\lambda\mu_i) \in \Phi_\times(\mathcal{X})$ for a complex number $\lambda \neq 0$ if and only if $A-\lambda(B_0-\mu_i) \in \Phi_\times(\mathcal{X})$, and $A \in \Phi_\times(\mathcal{X})$ if and only if $A - B_0 \in \Phi_\times(\mathcal{X})$; (b) $\pi_B(A,\lambda) \in \Phi_\times(\mathcal{X})$ for some $\lambda \neq 0$ implies $A - \lambda B \in \Phi_\times(\mathcal{X})$. The case of operator A such $\pi_B(A,\lambda)$ has SVEP, the single-valued extension property, or essential SVEP, at 0 is also considered.

1. Auxiliary Results

Let $\mathrm{Inv}_\ell(\mathcal{X})$ ($\mathrm{Inv}_\mathrm{r}(\mathcal{X})$) denote the class of operators $A \in \mathcal{B}[\mathcal{X}]$ which are left invertible (resp., right invertible). Let $\mathcal{T}$ denote the *Calkin homomorphism* $\mathcal{T} : \mathcal{B}[\mathcal{X}] \to \mathcal{B}[\mathcal{X}]/\mathcal{K}[\mathcal{X}]$. Then $A \in \mathcal{K}[\mathcal{X}]$ if and only if $\mathcal{T}(A) = 0$, $A \in \mathcal{R}[\mathcal{X}]$ if and only if $\mathcal{T}(A)$ is a quasinilpotent operator, and an $A \in \mathcal{B}[\mathcal{X}]$ is in $\Phi_\ell(\mathcal{X})$ (resp., $\Phi_r(\mathcal{X})$) if and only if $\mathcal{T}(A) \in \mathrm{Inv}_\ell(\mathcal{X})$ (resp., $\mathcal{T}(A) \in \mathrm{Inv}_\mathrm{r}(\mathcal{X})$). Let $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$. Then there exists a function $f \in \mathcal{H}_c(\sigma(B))$ such that $f(B) \in \mathcal{K}[\mathcal{X}]$, and hence such that $\mathcal{T}(f(B)) = f(\mathcal{T}(B)) = 0$. Since $f(z)$ has at best a finite number of zeros, there exists a polynomial $p(.)$ such that $f(\mathcal{T}(B)) = p(\mathcal{T}(B))g(\mathcal{T}(B)) = 0$, where the (holomorphic on $\sigma(B)$) function g satisfies the property that $g(z) \neq 0$ on $\sigma(B)$. But then $p(\mathcal{T}(B)) = 0$, and hence there exists a monic irreducible polynomial, *the minimal polynomial of* B, which divides every other polynomial $q(z)$ such that $q(\mathcal{T}(B)) = 0$. If we let $\pi_B(z) = \prod_{i=1}^m (z-\mu_i)^{\alpha_i}$ denote the minimal polynomial of B, then $\pi_B(B) \in \mathcal{K}[\mathcal{X}]$. In the case in which $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$, so that $f(B) \in \mathcal{R}[\mathcal{X}]$ for some $f \in \mathcal{H}_c(\sigma(B))$, $f(\mathcal{T}(B))$ is a quasinilpotent such that $f(\mathcal{T}(B)) = p(\mathcal{T}(B))g(\mathcal{T}(B))$ for some polynomial $p(.)$ such that $p(\mathcal{T}(B))$ is quasinilpotent and the function $g(.)$ is invertible. Once again there exists a minimal polynomial $\pi_B(.)$ of B such that $\pi_B(B) \in \mathcal{R}[\mathcal{X}]$. We have ([13, 11, 16]):

Proposition 1.1. *The following conditions are equivalent for operators $B \in \mathcal{B}[\mathcal{X}]$:*

(i) $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$ *(resp., $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$).*

(ii) *B is polynomially compact (resp., polynomially Riesz).*

(iii) *There exists a monic irreducible polynomial $\pi_B(z) = \prod_{i=1}^m (z-\mu_i)^{\alpha_i}$ (resp., $\pi_B(z) = \prod_{i=1}^m (z-\mu_i)$), the minimal polynomial of B, such that $\pi_B(B)$ is compact (resp., Riesz).*

If $f(B) \in \mathcal{K}[\mathcal{X}] \cup \mathcal{R}[\mathcal{X}]$ is such that (the Fredholm essential spectrum) $\sigma_e(f(B)) \neq \emptyset$, then (it follows from the considerations above that) there exists a finite subset $\{\mu_1, \mu_2, \cdots, \mu_m\}$ of the set of complex numbers $\mathbb{C}$ such that $f(\mu_i) = 0$ for all $1 \leq i \leq m$, and there exist disjoint countable subsets $S_i = \{\mu_{i_n}\} \subset \mathbb{C}$ such that μ_{i_n} converges to $\mu_i \in \mathcal{S}_i$ and $S_1 \cup S_2 \cup \cdots \cup S_m = \sigma(B)$. (Here either of the sets S_i may consist just of the singleton μ_i, and then *the quasinilpotent part* $H_0(B-\mu_i) = \{x \in \mathcal{X} : \lim_{n\to\infty} ||(B-\mu_i)^n x||^{\frac{1}{n}} = 0\}$ of $B - \mu_i$ is infinite dimensional.) Letting P_i denote the spectral projection associated with the spectral set S_i we then obtain spectral subspaces $\mathcal{X}_i$ of $\mathcal{X}$ and operators $B_i = B|_{\mathcal{X}_i}$ such that $\mathcal{X} = \oplus_{i=1}^m \mathcal{X}_i$, $B = \oplus_{i=1}^m B_i$ and $\sigma_e(B_i) = \{\mu_i\}$. Furthermore, each $(B_i - \mu_i)^{\alpha_i}$ is compact in the case in which $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$, and (since, for an operator $E \in \mathcal{B}[\mathcal{X}]$, $E^{\alpha_i} \in \mathcal{R}[\mathcal{X}]$ if and only if $E \in \mathcal{R}[\mathcal{X}]$) each $B_i - \mu_i$ is Riesz in the case in which $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$. We have:

Proposition 1.2. [8, 16] *If $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$ (resp., $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$), then there exists a finite subset $\{\mu_1, \mu_2, \cdots \mu_m\} \subset \mathbb{C}$, a subset $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ of positive integers, a decomposition $\mathcal{X} = \oplus_{i=1}^m \mathcal{X}_i$ of $\mathcal{X}$ into closed B-invariant subspaces and a decomposition $B = \oplus_{i=1}^m B_i$ of B such that each $(B_i - \mu_i)^{\alpha_i}$ is compact (resp., each $B_i - \mu_i$ is Riesz).*

2. Riesz Perturbations

Given operators $A, B \in \mathcal{B}[\mathcal{X}]$, let $\delta_{A,B} \in \mathcal{B}[\mathcal{B}[\mathcal{X}]]$ denote the generalized derivation $\delta_{A,B}(X) = AX - XB$, and let $\delta_{A,B}^n(X) = \delta_{A,B}^{n-1}(\delta_{A,B}(X))$. The operators A, B are said to be *quasinilpotent equivalent* if

$$\lim_{n\to\infty} ||\delta_{A,B}^n(I)||^{\frac{1}{n}} = \lim_{n\to\infty} ||\delta_{B,A}^n(I)||^{\frac{1}{n}} = 0.$$

The following proposition is well known (see [14, Proposition 3.4.11], [6, Theorem 3.1]).

Proposition 2.1. *If A, B are quasinilpotent equivalent operators, then $\sigma_\times(A) = \sigma_\times(B)$, where $\sigma_\times$ stands for either of the left spectrum, the right spectrum, the approximate point spectrum σ_a, the surjectivity spectrum σ_s and the spectrum σ.*

We assume in the following that if an operator $B \in \mathcal{B}[\mathcal{X}]$ is such that $B \in Holo^{-1}(\mathcal{K}[\mathcal{X}])$ or $Holo^{-1}(\mathcal{R}[\mathcal{X}])$, then it has the minimal polynomial function of Proposition 1.1, the Banach space $\mathcal{X}$ and the operator B have the decompositions $X = \oplus_{i=1}^m \mathcal{X}_i$ and $B = \oplus_{i=1}^m B_i$ of Proposition 1.2. *The operator $B_0 \in \mathcal{B}[\mathcal{X}]$ shall henceforth be defined by $B_0 = \bigoplus_{i=1}^m (B_i - \mu_i)$, where the scalars μ_i are as defined in Proposition 1.1.* Let $\text{Inv}_\times(\mathcal{X})$ denote operators $A \in \mathcal{B}[\mathcal{X}]$ which are either bounded below or surjective.

Given operators $A, B \in \mathcal{B}[\mathcal{X}]$, let $[A, B]$ denote *the commutator* $[A, B] = AB - BA$ of A and B. If $\Phi_\times(\mathcal{X})$ denotes either of $\Phi_\ell(\mathcal{X})$ or $\Phi_r(\mathcal{X})$ or $\Phi_\pm(\mathcal{X}) = \Phi_+(\mathcal{X}) \cup$

$\Phi_-(\mathcal{X})$, then the perturbation class of $\Phi_\times(\mathcal{X})$, $\mathrm{Ptrb}(\Phi_\times(\mathcal{X}))$, is the closed two sided ideal

$$\mathrm{Ptrb}(\Phi_\times(\mathcal{X})) = \{\mathrm{A} \in \mathcal{B}[\mathcal{X}] : \mathrm{A} + \mathrm{B} \in \Phi_\times(\mathcal{X}) \ \text{ for every } \ \mathrm{B} \in \Phi_\times(\mathcal{X})\}.$$

It is seen that

$$\mathrm{Ptrb}(\Phi_\ell(\mathcal{X})) = \mathrm{Ptrb}(\Phi_\mathrm{r}(\mathcal{X})) = \mathrm{Ptrb}(\Phi(\mathcal{X})) \supseteq \mathrm{Ptrb}(\Phi_+(\mathcal{X})) \cup \mathrm{Ptrb}(\Phi_-(\mathcal{X})).$$

Let $\mathcal{T}_p$ denote the homomorphism

$$\mathcal{T}_p : \mathcal{B}[\mathcal{X}] \to \mathcal{B}[\mathcal{X}]/\mathrm{Ptrb}(\Phi_\times(\mathcal{X}))$$

which is effected by the natural projection of the algebra $\mathcal{B}[\mathcal{X}]$ into the quotient algebra $\mathcal{B}[\mathcal{X}]/\mathrm{Ptrb}(\Phi_\times(\mathcal{X}))$. It is then clear that $[A,B] = AB - BA \in \mathrm{Ptrb}(\Phi_\times(\mathcal{X}))$ if and only if $\mathcal{T}_p(AB - BA) = \mathcal{T}_p(A)\mathcal{T}_p(B) - \mathcal{T}_p(B)\mathcal{T}_p(A) = 0$; furthermore, if the function $f \in \mathrm{Holo}^{-1}(\sigma(\mathrm{A}) \cup \sigma(\mathrm{B}))$, in particular if f is a polynomial, then $[A,B] \in \mathrm{Ptrb}(\Phi_\times(\mathcal{X}))$ implies $f(A)f(B) - f(B)f(A) \in \mathrm{Ptrb}(\Phi_\times(\mathcal{X}))$, and hence $\mathcal{T}_p(f(A)f(B) - f(B)f(A)) = 0$.

Theorem 2.1. *Let $A, B \in \mathcal{B}[\mathcal{X}]$ be such that $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$.*

(a) *If $\pi_B(A,\lambda) = \prod_{i=1}^m (A - \lambda\mu_i) \in \Phi_\times(\mathcal{X})$ for some complex number λ and $[A,B] \in \mathrm{Ptrb}(\Phi_\times(\mathcal{X}))$, then $A - \lambda B \in \Phi_\times(\mathcal{X})$ if $\lambda \neq 0$, and $A - B_0 \in \Phi_\times(\mathcal{X})$ whenever $\lambda = 0$.*

(b) *Suppose that $[A,B] = 0$.*

(i) *If $\lambda \neq 0$, then $\pi_B(A,\lambda) = \prod_{i=1}^m (A - \lambda\mu_i)^{\alpha_i} \in \Phi_\times(\mathcal{X})$ if and only if $A - \lambda B_0 - \lambda\mu_i \in \Phi_\times(\mathcal{X})$.*

(ii) *(Case $\lambda = 0$) $A \in \Phi_\times(\mathcal{X})$ if and only if $A - B_0 \in \Phi_\times(\mathcal{X})$.*

(c) *If $\lambda \neq 0$, $[A,B] = 0$ and $\pi_B(A,\lambda) \in \Phi_\times(\mathcal{X})$, then $A - \lambda B \in \Phi_\times(\mathcal{X})$.*

Proof. (a) Define the operators D, E and F by

$$D = E - F,\ E = \pi_B(A,\lambda) \ \text{ if } \ \lambda \neq 0 \ \text{ and } \ E = A^m \ \text{ if } \ \lambda = 0,$$
$$F = \lambda^m \pi_B(B) \ \text{ if } \ \lambda \neq 0 \ \text{ and } \ F = B_0^m \ \text{ if } \ \lambda = 0.$$

Then $F \in \mathcal{R}[\mathcal{X}]$, and the hypothesis that $[A,B] \in \mathrm{Ptrb}\Phi_\times(\mathcal{X})$ implies

$$\mathcal{T}_p[E,F] = \mathcal{T}_p(E)\mathcal{T}_p(F) - \mathcal{T}_p(F)\mathcal{T}_p(E) = 0.$$

The operator $\mathcal{T}_p(F)$ being quasinilpotent, we have

$$\begin{aligned}\delta^n_{\mathcal{T}_p(D),\mathcal{T}_p(E)}(I) &= \delta^{n-1}_{\mathcal{T}_p(D),\mathcal{T}_p(E)}((-1)\mathcal{T}_p(F)) \\ &= \cdots = (-1)^n\mathcal{T}_p(F)^n = \cdots = (-1)^n\delta^n_{\mathcal{T}_p(E),\mathcal{T}_p(D)}(I),\end{aligned}$$

and hence $\mathcal{T}_p(D)$ and $\mathcal{T}_p(E)$ are quasinilpotent equivalent. Since $E \in \Phi_\times(\mathcal{X})$,

$$\mathcal{T}_p(E) \in \mathrm{Inv}_\times(\mathcal{X}) \Longleftrightarrow \mathcal{T}_\mathrm{p}(\mathrm{D}) \in \mathrm{Inv}_\times(\mathcal{X}).$$

Again, since

$$\begin{aligned}\mathcal{T}_p(D) &= (\mathcal{T}_p(A) - \mathcal{T}_p(B))g(\mathcal{T}_p(A),\mathcal{T}_p(B),\lambda) \\ &= g(\mathcal{T}_p(A),\mathcal{T}_p(B),\lambda)(\mathcal{T}_p(A) - \lambda\mathcal{T}_p(B)) \ \text{ if } \ \lambda \neq 0,\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_p(D) &= \mathcal{T}_p(A)^m - \mathcal{T}_p(B_0)^m = (\mathcal{T}_p(A) - \mathcal{T}_p(B_0))g_1(\mathcal{T}_p(A),\mathcal{T}_p(B),\lambda) \\ &= g_1(\mathcal{T}_p(A),\mathcal{T}_p(B),\lambda)(\mathcal{T}_p(A) - \mathcal{T}_p(B_0)) \ \text{ if } \ \lambda = 0,\end{aligned}$$

it follows that

$$\mathcal{T}_p(A) - \lambda\mathcal{T}_p(B) \in \mathrm{Inv}_\times(\mathcal{X}) \text{ if } \lambda \neq 0 \text{ and}$$
$$\mathcal{T}_p(A) - \mathcal{T}_p(B_0) \in \mathrm{Inv}_\times(\mathcal{X}) \text{ if } \lambda = 0.$$

Since

$$\begin{aligned}
&A - \lambda B \text{ (resp., } A - B_0) \in \Phi_+(\mathcal{X}) \text{ if and only if}\\
&\mathcal{T}_p(A) - \lambda\mathcal{T}_p(B) \text{ (resp., } \mathcal{T}_p(A) - \mathcal{T}_p(B_0)) \text{ is bounded below and}\\
&A - \lambda B \text{ (resp., } A - B_0) \in \Phi_-(\mathcal{X}) \text{ if and only if}\\
&\mathcal{T}_p(A) - \lambda\mathcal{T}_p(B) \text{ (resp., } \mathcal{T}_p(A) - \mathcal{T}_p(B_0)) \text{ is surjective,}
\end{aligned}$$

the proof follows.

(b) The proof at places is similar to the one above, so we shall at points be brief. Let $\mathcal{T} : \mathcal{B}[\mathcal{X}] \to \mathcal{B}[\mathcal{X}]/\mathcal{K}[\mathcal{X}]$ denote the *Calkin homomorphism*. Suppose that $[A, B] = 0$. Letting $B = \oplus_{i=1}^m B_i$ with respect to the decomposition $\mathcal{X} = \oplus_{i=1}^m \mathcal{X}_i$ of $\mathcal{X}$, it is seen that A has a matrix representation $A = (A_{ij})_{i,j=1}^m$ such that

$$\begin{aligned}
&A_{ij}B_j = B_iA_{ij} \text{ for all } 1 \leq i, j \leq m\\
\iff\ &A_{ij}(B_j - \mu_i) = (B_i - \mu_i)A_{ij} \text{ for all } 1 \leq i, j \leq m.'
\end{aligned}$$

Here, the complex numbers μ_i, $1 \leq i \leq m$, are distinct, the operators $B_i - \mu_i$ being Riesz for all $1 \leq i \leq m$ and (since $\mu_i \notin \sigma(B_j)$ for all $1 \leq i \neq j \leq m$) the operator $\mathcal{T}(B_j - \mu_i)$ is invertible for all $1 \leq i \neq j \leq m$. Consequently,

$$\begin{aligned}
&\mathcal{T}(A_{ij})\mathcal{T}(B_j - \mu_i)^n = \mathcal{T}(B_i - \mu_i)^n\mathcal{T}(A_{ij})\\
\iff\ &\mathcal{T}(A_{ij}) = \mathcal{T}(B_j - \mu_i)^{-n}\mathcal{T}(B_i - \mu_i)^n\mathcal{T}(A_{ij}).
\end{aligned}$$

We have two possibilities: Either $\mathcal{T}(A_{ij}) \neq 0$ or $\mathcal{T}(A_{ij}) = 0$. If $\mathcal{T}(A_{ij}) \neq 0$, then (since $\mathcal{T}(B_i - \mu_i)$ is quasinilpotent)

$$\begin{aligned}
&||\mathcal{T}(A_{ij})|| \leq ||\mathcal{T}(A_{ij})||||\mathcal{T}(B_j - \mu_i)^{-1}||^n||\mathcal{T}(B_i - \mu_i)^n||\\
\Longrightarrow\ &1 \leq ||\mathcal{T}(B_j - \mu_i)^{-1}||\lim_{n\to\infty}||\mathcal{T}(B_i - \mu_i)^n||^{\frac{1}{n}} = 0.
\end{aligned}$$

This being a contradiction, we must have

$$\mathcal{T}(A) = \oplus_{i=1}^m\mathcal{T}(A_{ii}), \mathcal{T}(A_{ij}) = 0 \text{ and } [A_{ii}, B_i] = 0 \text{ for all } 1 \leq i \neq j \leq m.$$

Define the operators $M_j, N_j \in B[\mathcal{X}_j]$, $1 \leq j \leq m$, by

$$M_j = (A_{jj} - \lambda B_j) - \lambda(\mu_i - \mu_j),\ \ N_j = A_{jj} - \lambda\mu_i \text{ for all } 1 \leq i, j \leq m \text{ if } \lambda \neq 0,$$

and

$$M_j = A_{jj} - B_j + \mu_j,\ \ N_j = A_{jj} \text{ for all } 1 \leq j \leq m \text{ if } \lambda = 0.$$

Then the equivalences

$$\begin{aligned}
\pi_B(B) \in \mathcal{R}[\mathcal{X}] \iff\ &\prod_{i=1}^m(B - \mu_i) = \prod_{i=1}^m\{\oplus_{j=1}^m(B_j - \mu_i)\} \in \mathcal{R}[\mathcal{X}]\\
\iff\ &\prod_{i=1}^m(B_j - \mu_i) \in \mathcal{R}[\mathcal{X}_j] \text{ for all } 1 \leq j \leq m\\
\iff\ &B_j - \mu_j \in \mathcal{R}[\mathcal{X}_j] \text{ for all } 1 \leq j \leq m
\end{aligned}$$

and

$$\begin{aligned}\pi_B(A,\lambda)\in\Phi_\times(\mathcal{X}) &\iff \prod_{i=1}^m \mathcal{T}(A-\lambda\mu_i)=\prod_{i=1}^m\{\oplus_{j=1}^m\mathcal{T}(A_{jj}-\lambda\mu_i)\}\in \mathrm{Inv}_\times(\mathcal{X})\\ &\iff \prod_{i=1}^m \mathcal{T}(A_{jj}-\lambda\mu_i)=\mathcal{T}\{\prod_{i=1}^m(A_{jj}-\lambda\mu_i)\}\in \mathrm{Inv}_\times(\mathcal{X}_{\mathrm{j}})\\ &\qquad \text{for all } 1\le i,j\le m\\ &\iff \prod_{i=1}^m(A_{jj}-\lambda\mu_i)\in\Phi_\times(\mathcal{X}_j)\ \text{ for all } 1\le i,j\le m\\ &\iff A_{jj}-\lambda\mu_i\in\Phi_\times(\mathcal{X}_j)\ \text{ for all } 1\le i,j\le m\end{aligned}$$

imply that

$$\begin{aligned}\delta^n_{\mathcal{T}(M_j),\mathcal{T}(N_j)}(I_j) &= (-\lambda)\delta^{n-1}_{\mathcal{T}(M_j),\mathcal{T}(N_j)}\mathcal{T}(B_j-\mu_j)=\cdots=(-\lambda)^n\mathcal{T}(B_j-\mu_j)^n\\ &= \cdots=\delta^n_{\mathcal{T}(N_j),\mathcal{T}(M_j)}(I_j).\end{aligned}$$

This implies that the operators $\mathcal{T}(M_j)$ and $\mathcal{T}(N_j)$ are quasinilpotent equivalent, and hence

$$M_j\in\Phi_\times(\mathcal{X}_j)\iff N_j\in\Phi_\times(\mathcal{X}),\ \ 1\le j\le m.$$

Now if we define $B_0\in\mathcal{B}[\mathcal{X}]$ (as above) by $B_0=\oplus_{j=1}^m(B_j-\mu_j)$, then

$$\begin{aligned}\mathcal{T}(A-\lambda B_0-\lambda\mu_i) &= \oplus_{j=1}^m\{\mathcal{T}((A_{jj}-\lambda B_j)-\lambda(\mu_i-\mu_j))\}\in\mathrm{Inv}_\times(\mathcal{X})\\ &\qquad \text{for all } 1\le i\le m\\ &\iff \oplus_{j=1}^m\mathcal{T}(A_{jj}-\lambda\mu_i)\in\mathrm{Inv}_\times(\mathcal{X})\ \text{ for all } 1\le \mathrm{i}\le \mathrm{m}\\ &\iff \prod_{i=1}^m\{\oplus_{j=1}^m\mathcal{T}(A_{jj}-\lambda\mu_i)\}\in\mathrm{Inv}_\times(\mathcal{X})\\ &\qquad =\prod_{i=1}^m\mathcal{T}(A-\lambda\mu_i)\in\mathrm{Inv}_\times(\mathcal{X})\\ &\iff \pi_B(A,\lambda)\in\Phi_\times(\mathcal{X})\end{aligned}$$

if $\lambda\neq 0$, and

$$\begin{aligned}&\oplus_{j=1}^m\mathcal{T}(M_j)=\oplus_{j=1}^m\mathcal{T}(A_{jj}-B_j+\mu_j)=\mathcal{T}(A-B_0)\in\mathrm{Inv}_\times(\mathcal{X})\\ \iff\ &\oplus_{j=1}^m\mathcal{T}(N_j)=\oplus_{j=1}^m\mathcal{T}(A_{jj})=\mathcal{T}(\pi_B(A,0))\in\mathrm{Inv}_\times(\mathcal{X})\\ \iff\ &\pi_B(A,0)\in\Phi_\times(\mathcal{X})\end{aligned}$$

if $\lambda=0$.

(c) Let $\lambda\neq 0$. Choosing $i=j$ in

$$\pi_B(A,\lambda)\in\Phi_\times(\mathcal{X})\iff A-\lambda(\oplus_{j=1}^m(B_j-\mu_j+\mu_i)\in\Phi_\times(\mathcal{X})$$

for all $1\le i\le m$, it then follows that

$$\pi_B(A,\lambda)\in\Phi_\times(\mathcal{X})\Longrightarrow A-\lambda B\in\Phi_\times(\mathcal{X}).$$ □

Remark 2.1. (i) Some hypothesis of the type $[A,B]\in\mathrm{Prtb}\Phi_\times(\mathcal{X})$, or $[A,B]=0$, is essential to the validity of Theorem 2.1. To see this, consider operators A,B such that $\pi_B(A,\lambda)\in\Phi_\times(\mathcal{X})$ and $\pi_B(B)$ is compact. Then, since $\mathcal{T}_p(\pi_B(B))=0=\mathcal{T}(\pi_B(B))$, $\pi_B(A,\lambda)-\lambda^m\pi_B(B)\in\Phi_\times(\mathcal{X})\iff\pi_B(A,\lambda)\in\Phi_\times(\mathcal{X})$. This does not however imply $A - \lambda B$ (or, $A - B_0$ if $\lambda = 0$, or $A - \lambda B_0 - \mu_i$ if $\lambda \neq 0$) $\in \Phi_\times(\mathcal{X})$, as the following elementary example shows. Letting I denote the identity of $\mathcal{B}[\mathcal{X}]$, define the polynomially compact operator B (with minimal polynomial $\pi_B(z) = (z-1)^2$) by $B = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$, and let $A = \begin{pmatrix} 2I & 0 \\ I & 0 \end{pmatrix}$. Then, with $\lambda = 1$, $\pi_B(A, \lambda) = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix}$ is invertible (hence, Fredholm). However the operator $A - \lambda B$ (which satisfies $(A - \lambda B)^2 = 0$) is not even semi-Fredholm. Again, if we define A by $A = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix}$, then $(A - B_0)^2 = 0$ and $A - B_0$ is not semi-Fredholm. Observe that neither of the hypotheses $[A, B] = 0$ or $[A, B] \in \mathrm{Ptrb}(\Phi_\times(\mathcal{X})$ is satisfied.

(ii) Let A and B be the operators of Theorem 2.1, parts (b) and (c). Then $A - \lambda\mu_i \in \Phi_\times(\mathcal{X})$ if and only if $A_{jj} - \lambda\mu_i \in \Phi_\times(\mathcal{X}_j)$ for all $1 \leq j \leq m$ and $\mathcal{T}(A_{ij}) = 0$ for all $1 \leq i \neq j \leq m$. The conclusion $\mathcal{T}(A_{ij}) = 0$ for all $1 \leq i \neq j \leq m$ implies that the operator $A = [A_{ij}]_{1 \leq i,j \leq m}$ may be written as the sum $A = A_1 + A_0$, where $A_1 = \oplus_{j=1}^m A_{jj}$ and the compact (hence, Riesz) operator A_0 is defined by

$$A_0 = [A_{ij}]_{1 \leq i,j \leq m} \text{ with } A_{ii} = 0 \text{ for all } 1 \leq i \leq m.$$

Recalling that the sum of two commuting Riesz operators is a Riesz operator, it follows that the operators $\frac{1}{\lambda}A_0 - B_0$ (case $\lambda \neq 0$) and $A_0 - B_0$ (case $\lambda = 0$) are Riesz operators. It is now seen that the operators

$$A - \lambda\mu_i - \lambda B_0 = (A_1 - \lambda\mu_i) + \lambda(\frac{1}{\lambda}A_0 - B_0) \text{ and } A_1 - \lambda\mu_i \ (\lambda \neq 0),$$
$$A - B_0 = A_1 + (A_0 - B_0) \text{ and } A_1 \ (\lambda = 0)$$

are quasinilpotent equivalent. Hence

$$A_1 - \lambda\mu_i \in \Phi_\times(\mathcal{X}) \Longleftrightarrow A - \lambda\mu_i - \lambda B_0 \in \Phi_\times(\mathcal{X}), \ \lambda \neq 0$$

and

$$A \in \Phi_\times(\mathcal{X}) \Longleftrightarrow A - B_0 \in \Phi_\times(\mathcal{X}).$$

This provides an alternative to some of the argument used to prove parts (b) and (c) of Theorem 2.1.

Let $\lambda(t)$ denote a continuous function from a connected subset $\mathcal{I}$ of the reals into $\mathcal{C}$ such that $\lambda(t_1) = 0$ and $\lambda(t_2) = 1$ for some $t_1, t_2 \in \mathcal{I}$, $t_1 < t_2$. Then (the argument of the proof of Theorem 2.1 holds with λ replaced by $\lambda(t)$ and we have):

Corollary 2.1. *Let $A, B \in \mathcal{B}[\mathcal{X}]$ be such that $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$.*

(a) *If $\pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda(t)\mu_i) \in \Phi_\times(\mathcal{X})$ and $[A, B] \in \mathrm{Ptrb}(\Phi_\times(\mathcal{X}))$, then $A - \lambda(t)B \in \Phi_\times(\mathcal{X})$ for all $t \in [t_1, t_2]$.*

(b) *If A, B commute, then:*

(i) *$\pi_B(A, \lambda(t)) = \prod_{i=1}^m (A - \lambda(t)\mu_i) \in \Phi_\times(\mathcal{X})$ if and only if $A - \lambda(t)(B_0 + \mu_i) \in \Phi_\times(\mathcal{X})$, $1 \leq i \leq m$, for all $t \in [t_1, t_2]$;*

(ii) *$\pi_B(A, \lambda(t_1)) \in \Phi_\times(\mathcal{X})$ if and only if $A - B_0 \in \Phi_\times(\mathcal{X})$;*

(iii) *$\pi_B(A, \lambda(t)) \in \Phi_\times(\mathcal{X})$ implies $A - \lambda(t)B \in \Phi_\times(\mathcal{X})$ for all $t \in [t_1, t_2]$.*

Recalling the fact that "every locally constant function on a connected set is constant", it follows from the local constancy of the index function "ind" that

$\operatorname{ind}(A) = \operatorname{ind}(A-B) = \operatorname{ind}(A-\lambda(t)B)$ for all $t \in [t_1, t_2]$. In particular, if $A \in \Phi_\ell(\mathcal{X})$ (resp., $A \in \Phi_r(\mathcal{X})$), then $(A-\lambda(t)B)(\mathcal{X})$ (resp., $(A-\lambda(t)B)^{-1}(0)$) is complemented by a finite dimensional subspace if and only if $A(\mathcal{X})$ (resp., $A^{-1}(0)$) is complemented by a finite dimensional subspace.

3. Operators with SVEP

$A \in \mathcal{B}[\mathcal{X}]$ has *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0 for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only holomorphic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(T-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. *T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.* Operators with countable spectrum have SVEP: If $A \in \mathcal{R}[\mathcal{X}]$, then both A and (the conjugate operator) A^* have SVEP. It is known that $f(A)$, $A \in \mathcal{B}[\mathcal{X}]$ and $f \in H_c(\sigma(A))$, has SVEP at a point λ if and only if A has SVEP at every μ such that $f(\mu) = \lambda$ (see [1, Theorem 2.39] and [14]). If an $A \in \mathcal{B}[\mathcal{X}]$ has SVEP at a point λ, then SVEP for $B \in \mathcal{B}[\mathcal{X}]$ at λ does not transfer to $A + B$, even if A and B commute. However:

Proposition 3.1. [2, Theorem 0.3] *If A and B commute, and if $B \in \mathcal{R}[\mathcal{X}]$, then SVEP at λ for A implies SVEP for $A - B$ at λ.*

Recall that the *ascent* (resp., *descent*) of $A \in \mathcal{B}[\mathcal{X}]$, $\operatorname{asc}(A)$ (resp., $\operatorname{dsc}(A)$), is the least non-negative integer n such that $A^{-n}(0) = A^{-(n+1)}(0)$ (resp., $A^n(\mathcal{X}) = A^{n+1}(\mathcal{X})$); if no such integer exists, then $\operatorname{asc}(A) = \infty$ (resp., $\operatorname{dsc}(A) = \infty$). Finite ascent (resp., descent) at a point λ for A implies $\operatorname{ind}(A-\lambda) \leq 0$ and A has SVEP at λ (resp., $\operatorname{ind}(A-\lambda) \geq 0$ and A^* has SVEP at λ); conversely, if $A - \lambda \in \Phi_\times(\mathcal{X})$ (resp., $A^* - \lambda \in \Phi_\times(\mathcal{X})$) has SVEP at 0, then $\operatorname{asc}(A-\lambda) < \infty$ and $0 \in \operatorname{iso}\sigma_a(A)$ (resp., $\operatorname{dsc}(A-\lambda) < \infty$ and $0 \in \operatorname{iso}\sigma_s(A)$) [1, Theorems 3.16, 3.17, 3.23, 3.27]. The operator A is *upper Browder* (resp., *lower Browder, left Browder, right Browder, or (simply) Browder*) if it is upper Fredholm with $\operatorname{asc}(A) < \infty$ (resp., lower Browder with $\operatorname{dsc}(A) < \infty$, left Browder with $\operatorname{asc}(A) < \infty$, right Browder with $\operatorname{dsc}(A) < \infty$, Fredholm with $\operatorname{asc}(A) = \operatorname{dsc}(A) < \infty$). Let $A \in \times-$Browder denote that A is one of upper Browder, lower Browder, left Browder, right Browder or (simply) Browder. It is well known, see [9, Theorem 7.92.] or [6, Proposition 2.2], that *if $A, B \in \mathcal{B}[\mathcal{X}]$ are commuting operators, then $AB \in \times-$Browder if and only if $A, B \in \times-$Browder.* If an operator $A \in \{\Phi_+(\mathcal{X}) \cup \Phi_\ell(\mathcal{X})\}$ (resp., $A \in \{\Phi_-(\mathcal{X}) \cup \Phi_r(\mathcal{X})\}$ and A^*) has SVEP at 0, then A is upper or left (resp., lower or right) Browder [1, Theorem 3.52]. As before, the operator $B_0 \in \mathcal{B}[\mathcal{X}]$ is defined by $B_0 = \oplus_{j=1}^m (B_j - \mu_j)$.

The following theorem generalizes [6, Theorem 4.1].

Theorem 3.1. *Let $A, B \in \mathcal{B}[\mathcal{X}]$ be such that $[A, B] = 0$, $\pi_B(B) = \prod_{i=1}^m (B - \mu_i) \in \mathcal{R}[\mathcal{X}]$ and $\pi_B(A, \lambda) = \prod_{i=1}^m (A - \lambda\mu_i) \in \Phi_\times(\mathcal{X})$ for some complex number λ. Then:*

(a) *$A \in \times-$Browder if and only if $A - B_0 \in \times-$Browder.*

(b) (i) *$\pi_B(A, \lambda) \in \times-$Browder implies $A - \lambda B \in \times-$Browder, and* (ii) *$\pi_B(A, \lambda) \in \times-$Browder if and only if $A - \lambda B_0 - \lambda\mu_i \in \times-$Browder for all $1 \leq i \leq m$.*

(c) *If $A \in \{\Phi_+(\mathcal{X}) \cup \Phi_\ell(\mathcal{X})\}$ has SVEP at 0 (resp., $A \in \{\Phi_-(\mathcal{X}) \cup \Phi_r(\mathcal{X})\}$ and A^* has SVEP at 0), then $A - \lambda B$ is upper or, respectively, left (resp., lower or, respectively, right) Browder.*

Proof. We consider the case $\times-$Browder = upper Browder or left Browder only (thus $\times$ in $\Phi_\times$ shall stand for upper or left); the proof for the other cases is similar.

(a) The operator $B_0 = \oplus_{i=1}^m (B_i - \mu_i)$ being the direct sum of Riesz operators is a Riesz operator. Since A commutes with B_0, $A - B_0$ has SVEP at 0 if and only if A has SVEP at 0. Again, by Theorem 2.1(b.ii), $A - B_0 \in \Phi_\times(\mathcal{X})$ if and only if $A \in \Phi_\times(\mathcal{X})$. Hence, since an operator T is $\times-$Browder if and only if $T \in \Phi_\times(\mathcal{X})$ and T has SVEP at 0 [1, Theorem 3.52], $A - B_0 \in \times-$ Browder if and only if $A \in \times-$Browder.

(b.i) The hypothesis $\pi_B(A, \lambda) \in \times-$Browder implies $A - \lambda\mu_i \in \times-$Browder if and only if $A - \lambda\mu_i \in \Phi_\times(\mathcal{X})$ and $A - \lambda\mu_i$ has SVEP at 0. Since $\pi_B(B) = \prod_{i=1}^m (B - \mu_i)$ is Riesz, there an integer i, $1 \le i \le m$, such that $\lambda(B - \mu_i)$ is Riesz (and commutes with $A - \lambda\mu_i$). Hence $A - \lambda B = (A - \lambda\mu_i) - (B - \lambda\mu_i)$ has SVEP at 0. Since $A - \lambda B \in \Phi_\times(\mathcal{X})$ by Theorem 2.1(c), $A - \lambda B \in \times-$Browder.

(b.ii) The case $\lambda = 0$ being evident, we consider $\lambda \neq 0$. It is clear from Theorem 2.1(b.i) that

$$\pi_B(A, \lambda) \in \Phi_\times(\mathcal{X}) \Longleftrightarrow A - \lambda B - \lambda\mu_i \in \Phi_\times(\mathcal{X}).$$

Since

$$\begin{aligned} \pi_B(A, \lambda) \in \times-\text{Browder} &\Longleftrightarrow A - \lambda\mu_i \in \times-\text{Browder for all } 1 \le i \le m \\ &\Longleftrightarrow A - \lambda\mu_i \in \Phi_\times(\mathcal{X}), A - \lambda\mu_i \text{ has SVEP at } 0 \\ &\qquad \text{for all } 1 \le i \le m. \end{aligned}$$

The operator B_0 being a Riesz operator which commutes with $A - \lambda\mu_i$, it follows that $A - \lambda\mu_i - \lambda B_0$ has SVEP at 0 if and only if $A - \lambda\mu_i$ has SVEP at 0. Hence

$$\pi_B(A, \lambda) \in \times-\text{Browder} \Longleftrightarrow A - \lambda B_0 - \lambda\mu_i \in \times-\text{Browder}.$$

(c) Recall from above that if an operator $A \in \Phi_\times(\mathcal{X})$ has SVEP at 0, then $0 \in \text{iso}\sigma_a(A)$. Since $\sigma_a(A - \lambda\mu_i) \subset \sigma_a(A) - \{\lambda\mu_i\}$, it follows from our hypotheses that (at worst) $\lambda\mu_i \in \text{iso}\sigma_a(A)$ for all $1 \le i \le m$. Hence $A - \lambda\mu_i$ has SVEP at 0. As seen above, $A - \lambda B \in \Phi_\times(\mathcal{X})$. Hence, since the operator $B - \mu_i$ is Riesz and commutes with $A - \lambda\mu_i$, $A - \lambda B_i = (A - \lambda\mu_i) - \lambda(B_i - \mu_i)$ has SVEP at 0. This, [1, Theorem 3.52], implies that $A - \lambda B \in \times-$Browder. □

Remark 3.1. An alternative argument proving Theorem 3.1(b.i) goes as follows. If $\times =$ upper or left, then the hypotheses imply that $\pi_B(A, \lambda)$ has SVEP at 0 and the Riesz operator $\pi_B(B)$ commutes with $\pi_B(A, \lambda)$. Hence $\pi_B(A, \lambda) - \lambda^m \pi_B(B)$ has SVEP at 0. Already we know from (the proof of) Theorem 2.1 that $\pi_B(A, \lambda) - \lambda^m \pi_B(B) \in \Phi_\times(\mathcal{X})$, where $\Phi_\times(\mathcal{X}) = \Phi_+(\mathcal{X}) \cup \Phi_\ell(\mathcal{X})$. Hence $\pi_B(A, \lambda) - \lambda^m \pi_B(B) = (A - \lambda B) g(A, B, \lambda) = g(A, B, \lambda)(A - \lambda B)$ is upper or (resp.) left Browder. This implies $A - \lambda B$ is upper or (resp.) left Browder.

Essential SVEP. Let $\mathcal{T}_q : \mathcal{B}[\mathcal{X}] \to \mathcal{B}[\mathcal{X}_q]$, $\mathcal{X}_q = \ell^\infty(\mathcal{X})/m(\mathcal{X})$, denote the homomorphism effecting the'essential enlargement $A \to \mathcal{T}_q(A) = A_q$" of [4] (and [15, Theorems 17.6 and 17.9]). Then $A \in \mathcal{B}[\mathcal{X}]$ is upper semi-Fredholm (lower semi-Fredholm), $A \in \Phi_+(\mathcal{X})$ (resp., $A \in \Phi_-(\mathcal{X})$), if and only if A_q is bounded below (resp., A_q is surjective); $A_q = 0$ for an operator A if and only if A is compact, and if $A \in \mathcal{R}[\mathcal{X}]$, then A_q is a quasinilpotent. Recall from Theorem 2.1(b.ii) and (c) that if $A, B \in \mathcal{B}[\mathcal{X}]$ are such that $[A, B] = 0$, $\pi_B(B) = \prod_{i=1}^m (B - \mu_i) \in \mathcal{R}[\mathcal{X}]$

and $\pi_B(A,\lambda) = \prod_{i=1}^{m}(A-\lambda\mu_i) \in \Phi_{\pm}(\mathcal{X})$, then $A-\lambda B \in \Phi_{\pm}(\mathcal{X})$ if $\lambda \neq 0$ and $A-B_0 \in \Phi_{\pm}(\mathcal{X})$ if $\lambda = 0$. If we now assume that $\pi_B(A,\lambda) \in \Phi_-(\mathcal{X})$ (resp., the conjugate operator $\pi_B(A,\lambda)^* \in \Phi_-(\mathcal{X})$), $\lambda \neq 0$, has SVEP at 0, then $A-\lambda B \in \Phi(\mathcal{X})$ is inner regular. Again, if we assume $\lambda = 0$ and $A \in \Phi_-(\mathcal{X})$ (resp., $A^* \in \Phi_-(\mathcal{X})$) has SVEP at 0, then $A-B_0 \in \Phi(\mathcal{X})$ is inner regular. SVEP for an operator neither implies nor is implied by SVEP for its image under the homomorphisms $\mathcal{T}_q$ [3, 2.9 Remark]: We say in the following that A has *essential SVEP* at a point λ if $A_q = \mathcal{T}_q(A)$ has SVEP at λ. The following corollary says that a result similar to the one above on the inner regularity of $A-\lambda B$ and $A-B_0$ holds with the hypotheses on SVEP replaced by hypotheses on essential SVEP.

Corollary 3.1. *Let $A,B \in \mathcal{B}[\mathcal{X}]$ be such that $[A,B]=0$, $\pi_B(B) = \prod_{i=1}^{m}(B-\mu_i) \in \mathcal{R}[\mathcal{X}]$, $\pi_B(A,\lambda)$ has essential SVEP at 0 whenever $\pi_B(A,\lambda) \in \Phi_-(\mathcal{X})$ and $\pi_B(A,\lambda)^*$ has essential SVEP at 0 whenever $\pi_B(A,\lambda) \in \Phi_+(\mathcal{X})$, then $A-\lambda B \in \Phi(\mathcal{X})$ if $\lambda \neq 0$ and $A-B_0 \in \Phi(\mathcal{X})$ if $\lambda = 0$.*

Proof. We consider the case in which $\pi_B(A,\lambda) \in \Phi_+(\mathcal{X})$ and $\pi_B(A,\lambda)^*$ has essential SVEP at 0: The proof for the other case is similar. Arguing as in the proof of Theorem 2.1, the hypotheses $[A,B]=0$, $\pi_B(B) \in \mathcal{R}[\mathcal{X}]$ and $\pi_B(A,\lambda) \in \Phi_+(\mathcal{X})$ imply that if $\lambda \neq 0$, then

$$\begin{aligned} & A-\lambda\mu_i \ \text{ and } \ A-\lambda B \in \Phi_+(\mathcal{X}) \ \text{ for all } \ 1 \leq i \leq m \\ \iff \ & T_q(A-\lambda\mu_i) \ \text{ and } \ T_q(A-\lambda B) \ \text{ are bounded below for all } \ 1 \leq i \leq m \end{aligned}$$

and if $\lambda = 0$, then

$$A \ \text{ and } \ A-B_0 \in \Phi_+(\mathcal{X}) \iff T_q(A) \ \text{ and } \ T_q(A-B_0) \ \text{ are bounded below.}$$

Since $T_q(A-\lambda\mu_i)$ is bounded below for all $\leq i \leq m$ implies $\pi_B(A,\lambda)$ is bounded below, it follows from the hypothesis $T_q(\pi_B(A,\lambda)^*)$ has SVEP that

$$T_q(\pi_B(A,\lambda)) \ \text{ is invertible } \iff T_q(A-\lambda\mu_i) \ \text{ is invertible for all } \ 1 \leq i \leq m$$

[1, Corollary 2.24]. Letting A and B have the representations $A = [A_{ij}]_{1\leq i,j\leq m} \in B(\oplus_{j=1}^{m}\mathcal{X}_j)$ and $B = \oplus_{j=1}^{m} B_j \in B(\oplus_{j=1}^{m}\mathcal{X}_j)$ (as in the proof of Theorem 2.1), this implies that $T_q(A_{jj}-\lambda\mu_j)$ is invertible, and $T_q(B_j-\mu_j)$ is quasinilpotent, for all $1 \leq j \leq m$. Since the operators $T_q(A_{jj}-\lambda\mu_j)$ and $T_q(B_j-\mu_j)$ commute, $\sigma(T_q(A_{jj}-\lambda B_j)) \subset \sigma(T_q(A_{jj}-\lambda\mu_j))-\{0\}$ and $\sigma(A_{jj}-B_j+\mu_j) \subset \sigma(T_q(A_{jj}))-\{0\}$ for all $1 \leq j \leq m$. Hence the operators $T_q(A_{jj}-\lambda B_j)$ and $T_q(A_{jj}-B_j+\mu_j)$ are invertible for all $1 \leq j \leq m$. But then

$$T_q(A-\lambda B) = T_q\{\oplus_{j=1}^{m}(A_{jj}-\lambda B_j)\} \ \text{ invertible } \iff A-\lambda B \in \Phi(\mathcal{X})$$

and

$$T_q(A-B_0) = T_q\{\oplus_{j=1}^{m}(A_{jj}-B_j+\mu_j)\} \ \text{ invertible } \iff A-B_0 \in \Phi(\mathcal{X}).$$

This completes the proof. □

4. A Perturbed Inner Regular Operator

If $A \in \Phi_\times(\mathcal{X})$, $\Phi_\times = \Phi_\ell$ or Φ_r, then A has an inner generalized inverse, which we shall denote by $A^\dagger$ in the following. Clearly, the operator $AA^\dagger$ is (then) a projection from $\mathcal{X}$ onto $A(\mathcal{X})$, and $I-A^\dagger A$ is a projection from $\mathcal{X}$ onto $A^{-1}(0)$. Let N denote a complement of $A(\mathcal{X})$ and let M denote a complement of $A^{-1}(0)$. Then $A: M \oplus A^{-1}(0) \to A(\mathcal{X}) \oplus N$ has a matrix $A = A_1 \oplus 0$, where $A_1 \in \mathcal{B}[M, A(\mathcal{X})]$ is invertible. If $A^\dagger$ is any generalized inverse of A such that $A^\dagger A(\mathcal{X}) = M$ and $(AA^\dagger)^{-1}(0) = N$, then $A^{\dagger}_{M,N,E} = A^{\dagger} : A(\mathcal{X}) \oplus N \to M \oplus A^{-1}(0)$ has the form $A^{\dagger}_{M,N,E} = A_1^{-1} \oplus E$ for some arbitrary $E \in \mathcal{B}[N, A^{-1}(0)]$ [7, Page 37]. Now let $A, B \in \mathcal{B}[\mathcal{X}]$ be such that $B \in Holo^{-1}(\mathcal{R}[\mathcal{X}])$ (with minimal polynomial $\pi_B(z)$, defined as in Theorem 2.1), $AB - BA \in \mathrm{Ptrb}(\Phi_\ell(\mathcal{X}))$ and $\pi_B(A, \lambda) = \prod_{i=1}^{m} (A - \lambda\mu_i) \in \Phi_\ell(\mathcal{X})$ for some scalar λ. Then the operators $A - \lambda B$ if $\lambda \neq 0$ and $A - B_0$ if $\lambda = 0$ (with the operator B_0 as earlier defined) are in $\Phi_\ell(\mathcal{X})$. Letting S denote either of the operators $A - \lambda B$ and $A - B_0$, it then follows that S has an inner generalized inverse $S^{\dagger}$. In general $A(\mathcal{X})$ and $S(\mathcal{X})$, also $A^{-1}(0)$ and $S^{-1}(0)$, are quite distinct. However:

Theorem 4.1. *If $AA^{\dagger} = SS^{\dagger}$ and $A^{\dagger}A = S^{\dagger}S$, then A and S have the same range and the same null space, and $S^{\dagger}$ has a representation*

$$S^{\dagger} = (I - \lambda A^{\dagger}_{N,M,E} B)^{-1} A^{\dagger}_{N,M,F} \quad \text{if } \lambda \neq 0, \quad \text{and}$$
$$S^{\dagger} = (I - A^{\dagger}_{N,M,E} B_0)^{-1} A^{\dagger}_{N,M,F} \quad \text{if } \lambda = 0.$$

Here N is a complement of $A(\mathcal{X})$, M is a complement of $A^{-1}(0)$ and $E, F \in \mathcal{B}[N, A^{-1}(0)]$ are arbitrary.

Proof. If $AA^{\dagger} = SS^{\dagger}$ and $A^{\dagger}A = S^{\dagger}S$, then

$$S(\mathcal{X}) = SS^{\dagger}(\mathcal{X}) = AA^{\dagger}(\mathcal{X}) = A(\mathcal{X}), \qquad \text{and}$$
$$S^{-1}(0) = (S^{\dagger}S)^{-1}(0) = (A^{\dagger}A)^{-1}(0) = A^{-1}(0).$$

Now choose the subspaces N, M as above. For $A_1 = A|_M$, $S_1 = S|_M$ and every $E \in \mathcal{B}[N, A^{-1}(0)]$, if $\lambda \neq 0$, then the operator

$$\begin{aligned} I - \lambda A^{\dagger}_{N,M,E} B &= I + A^{\dagger}_{N,M,E}(S - A) \\ &= I + \begin{pmatrix} A_1^{-1} & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} S_1 - A_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1^{-1}S_1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

from $M \oplus A^{-1}(0)$ into $A(\mathcal{X}) \oplus N$ is invertible with the inverse satisfying

$$(I + A^{\dagger}_{N,M,E}(S - A))^{-1} A^{\dagger}_{N,M,F} = \begin{pmatrix} S_1^{-1}A_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0 \\ 0 & F \end{pmatrix} = \begin{pmatrix} S_1^{-1} & 0 \\ 0 & F \end{pmatrix}$$

for every operator $F \in \mathcal{B}[N, A^{-1}(0)]$. Again, if $\lambda = 0$, then

$$I - \lambda A^{\dagger}_{N,M,E} B_0 = I + A^{\dagger}_{N,M,E}(S - A) = \begin{pmatrix} A_1^{-1}S_1 & 0 \\ 0 & 1 \end{pmatrix}$$

from $M \oplus A^{-1}(0)$ into $A(\mathcal{X}) \oplus N$ is invertible with the inverse (as before) satisfying

$$(I + A^{\dagger}_{N,M,E}(S - A))^{-1} A^{\dagger}_{N,M,F} = \begin{pmatrix} S_1^{-1}A_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0 \\ 0 & F \end{pmatrix} = \begin{pmatrix} S_1^{-1} & 0 \\ 0 & F \end{pmatrix}$$

for every operator $F \in \mathcal{B}[N, A^{-1}(0)]$. Evidently, $SS^{\dagger}S = S$, where $S^{\dagger} = (I + A^{\dagger}_{N,M,E}(S - A))^{-1} A^{\dagger}_{N,M.F}$. □

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