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Uniform Stability for Time-Varying Infinite-Dimensional Discrete Linear Systems

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Stability for time-varying discrete linear systems in a Banach space is investigated. On the one hand is established a fairly complete collection of necessary and sufficient conditions for uniform asymptotic equistability for input-free systems. This includes uniform and strong power equistability, and uniform and strong ℓ_p-equistability, among other technical conditions which also play an essential role in stability theory. On the other hand, it is shown that uniform asymptotic equistability for input-free systems is equivalent to each of the following concepts of uniform stability for forced systems: ℓ_p-input ℓ_p-state, $\mathfrak{c}_0$-input $\mathfrak{c}_0$-state, bounded-input bounded-state, ℓ_p-input bounded-state (with $p > 1$), $\mathfrak{c}_0$-input bounded-state, and convergent-input bounded-state; these are also equivalent to their nonuniform counterparts. For time-varying convergent systems, the above is also equivalent to convergent-input convergent-state stability. The proofs presented here are all 'elementary' in the sense that they are based essentially only on the Banach–Steinhaus theorem.

Key words. Discrete-time systems, linear systems, infinite-dimensional systems, stability theory, time-varying systems.

1. Introduction

CONSIDER the class of infinite-dimensional discrete dynamical systems, operating in a deterministic environment, whose models are described by a linear difference equation evolving in a Banach space. Such a class can be naturally split into four subclasses, according to whether the models are homogeneous (i.e. input-free systems) or inhomogeneous (i.e. forced systems) on the one hand, and autonomous (i.e. time-invariant systems) or nonautonomous (i.e. time-varying systems) on the other hand. These comprise the whole class of systems that we shall be dealing with in this paper. Thus we omit, from now on, the qualifications infinite-dimensional, discrete, linear, and deterministic, since they will be implicitly understood.

The stability problem (mainly strong and uniform asymptotic stability) for input-free time-invariant systems has been investigated by several authors (e.g. see [3, 5, 6, 8, 12]). A few results for forced time-invariant systems have also been considered in the literature (e.g. see [3, 7, 12]). Some interesting results on stability for time-varying systems, which comprise the central theme of this paper, have recently appeared in the literature. For input-free systems, the relationship between weak, strong, and uniform power equistability, as well as between strong and uniform ℓ_p-equistability, was analysed in [11]; and further results on strong and uniform asymptotic equistability were presented in [9]. For forced systems, it was shown in [10] that ℓ_p-input bounded-state stability (with $p > 1$) implies uniform power equistability under the assumption of uniform equicontrollability. As far as the stability problem for continuous-time systems is concerned, see e.g. the references in [11].

In this paper, we shall be dealing with the uniform-stability problem for each of the four subclasses described above, with emphasis in forced time-varying systems. In Section 2, we present a fairly complete collection of necessary and sufficient conditions for uniform asymptotic equistability. Section 3 is concerned with a discussion of the results obtained in Section 2. These two sections deal with input-free time-varying systems. Their purpose is fourfold. They contain the auxiliary results that will be needed in the sequel, survey the previous results, introduce further new results, and present 'elementary' proofs for some known results which were originally established by 'nonelementary' means. Sections 4 and 5 investigate the stability problem for forced time-varying systems. The particular case of convergent time-varying systems is also considered.

The notation used throughout this paper is summarized as follows. $\mathbb{N}$ will denote the nonnegative integers (i.e. including zero). $\mathcal{X}$ will denote a Banach space and $\mathcal{B}[\mathcal{X}]$ the Banach algebra of all bounded linear operators of $\mathcal{X}$ into itself. We shall use $\|\bullet\|$ to denote both the norm in $\mathcal{X}$ and the induced uniform norm in $\mathcal{B}[\mathcal{X}]$. As usual, $\ell_p(\mathcal{X})$ (for any real number $p \geq 1$), $\mathfrak{c}_0(\mathcal{X})$, $\mathfrak{c}(\mathcal{X})$, and $\ell_\infty(\mathcal{X})$ (with their standard norms $\|\bullet\|_p$, $\|\bullet\|_\infty$, $\|\bullet\|_\infty$, and $\|\bullet\|_\infty$, respectively) will stand for the Banach spaces of all $\mathcal{X}$-valued sequences $\boldsymbol{x} = \big(x(i) : i \in \mathbb{N}\big) \in \mathcal{X}^{\mathbb{N}}$ such that

$$\|\boldsymbol{x}\|_p = \Big(\sum_{i=0}^{\infty} \|x(i)\|^p\Big)^{1/p} < \infty,$$

$\lim_{i\to\infty} \|x(i)\| = 0$, $\lim_{i\to\infty} \|x(i) - x\| = 0$ for some $x \in \mathcal{X}$, and

$$\|\boldsymbol{x}\|_\infty = \sup_{i\geq 0} \|x(i)\| < \infty,$$

respectively, so that $\ell_p(\mathcal{X}) \subset \mathfrak{c}_0(\mathcal{X}) \subset \mathfrak{c}(\mathcal{X}) \subset \ell_\infty(\mathcal{X})$. Given a sequence $\big(A(k) : k \in \mathbb{N}\big)$ of operators in $\mathcal{B}[\mathcal{X}]$, set $\Phi(k, k) = I$ (the identity in $\mathcal{B}[\mathcal{X}]$) for every $k \geq 0$, and

$$\Phi(k + l, k) = \prod_{j=k}^{k+l-1} A(j) = A(k + l - 1) \cdots A(k)$$

for every $l \geq 1$ and $k \geq 0$, so that

$$\Phi(k + l + m, k) = \Phi(k + l + m, k + l)\Phi(k + l, k)$$

for every $k,l,m \geq 0$. The double sequence $\big(\Phi(k + l, k) : k,l \in \mathbb{N}\big) \in \mathcal{B}[\mathcal{X}]^{\mathbb{N}^2}$ will be referred to as the *evolution operator process* associated with $\big(A(k) : k \in \mathbb{N}\big)$.

2. Input-free systems

Given any integer $k \geqslant 0$ and an arbitrary $x \in \mathcal{X}$, consider the sequence $(x(l) : l \in \mathbb{N}) \in \mathcal{X}^{\mathbb{N}}$ recursively defined by the following nonautonomous homogeneous difference equation.

(1a) $$x(l+1) = \Lambda(k+l)x(l), \qquad x(0) = x \in \mathcal{X},$$

whose solution is

(1b) $$x(l) = \Phi(k+l, k)x \quad \forall\, l \geqslant 0,$$

where $(\Phi(k+l, k) : k,l \in \mathbb{N})$ is the evolution operator process associated with the operator sequence $(\Lambda(k) : k \in \mathbb{N}) \in \mathcal{B}[\mathcal{X}]^{\mathbb{N}}$. The purpose of this section is to present, in a unified way, several necessary and sufficient conditions for uniform asymptotic equistability.

DEFINITION 1. The model (1), or equivalently the operator sequence $(\Lambda(k) : k \in \mathbb{N})$, is *uniformly asymptotically equistable* if, for each $\varepsilon > 0$, there exists an integer $l_\varepsilon \geqslant 0$ such that

$$l \geqslant l_\varepsilon \quad \Rightarrow \quad \|\Phi(k+l, k)x\| \leqslant \varepsilon\, \|x\| \quad \forall\, k \geqslant 0 \quad \forall\, x \in \mathcal{X},$$

or equivalently if $\sup_{k \geqslant 0} \|\Phi(k+l, k)\| \to 0$ as $l \to \infty$.

THEOREM 1. *Consider the model* (1). *The following assertions are equivalent.*

(A) $\lim_{l\to\infty} \sup_{k\geqslant 0} \|\Phi(k+l, k)\| = 0$.

(B) $\sup_{k\geqslant 0} \|\Phi(k+l, k)\| < 1 \quad \forall\, l \geqslant l_1$ *for some* $l_1 \geqslant 1$.

(C) $\sup_{k\geqslant 0} \|\Lambda(k)\| < \infty$ *and* $\sup_{k\geqslant 0} \|\Phi(k+l_0, k)\| < 1$ *for some* $l_0 \geqslant 1$.

(D) $\limsup_{l\to\infty} \sup_{k\geqslant 0} \|\Phi(k+l, k)\|^{1/l} < 1$.

(E) *There exist real constants* $\gamma \geqslant 1$ *and* $\alpha \in (0, 1)$ *such that*

$$\|\Phi(k+l, k)\| \leqslant \gamma\alpha^l \quad \forall\, k,l \geqslant 0.$$

(F) *For every* $p > 0$, *there exists a positive number* σ_p *such that*

$$\sum_{l=0}^{\infty} \|\Phi(k+l, k)\|^p \leqslant \sigma_p \quad \forall\, k \geqslant 0.$$

(G) *For every* $p > 0$, *there exists a positive number* σ_p *such that*

$$\sum_{l=0}^{\infty} \|\Phi(k+l, k)x\|^p \leqslant \sigma_p\, \|x\|^p \quad \forall\, k \geqslant 0 \quad \forall\, x \in \mathcal{X}.$$

(H) *For some* $q > 0$, *there exists a positive number* σ_q *such that*

$$\sum_{l=0}^{\infty} \|\Phi(k+l, k)\|^q \leqslant \sigma_q \quad \forall\, k \geqslant 0.$$

(I) *For some* $q > 0$, *there exists a positive number* σ_q *such that*

$$\sum_{l=0}^{\infty} \|\Phi(k+l, k)x\|^q \leqslant \sigma_q\, \|x\|^q \quad \forall\, k \geqslant 0 \quad \forall\, x \in \mathcal{X}.$$

(J) *For every* $p > 0$, *there exists a positive number* μ_p *such that*

$$\|\Phi(k+l, k)\|^p \leqslant (l+1)^{-1}\mu_p \quad \forall\, k,l \geqslant 0.$$

(K) *For some* $q > 0$, *there exists a positive number* μ_q *such that*

$$\|\Phi(k+l, k)\|^q \leqslant (l+1)^{-1}\mu_q \quad \forall\, k,l \geqslant 0.$$

(L) *For every* $p > 0$, *there exists a positive number* ρ_p *such that*

$$\sum_{m=0}^{l} \|\Phi(k+l, k+m)\|^p \leqslant \rho_p \quad \forall\, k,l \geqslant 0.$$

(M) *For some* $q > 0$, *there exists a positive number* ρ_q *such that*

$$\sum_{m=0}^{l} \|\Phi(k+l, k+m)\|^q \leqslant \rho_q \quad \forall\, k,l \geqslant 0.$$

(N) $\lim_{n\to\infty} \sup_{k\geqslant 0} \sup_{\nu\geqslant 0} \|\sum_{l=n}^{n+\nu} \Phi(k+l, k)\| = 0.$

Proof. To begin, let us remark that it is implicitly assumed in (A), (B), and (D) that the underlying nonnegative sequence $(\sup_{k\geqslant 0} \|\Phi(k+l, k)\| : l \in \mathbb{N})$ is well defined, i.e. $\sup_{k\geqslant 0} \|\Phi(k+l, k)\| < \infty$ for every $l \geqslant 0$, so that (for $l=1$) $\sup_{k\geqslant 0} \|A(k)\| < \infty$. Now consider the following auxiliary assertions.

(O) There exist real functionals $\gamma(\bullet), \alpha(\bullet) : \mathcal{X} \to \mathbb{R}$ such that $\gamma(x) \geqslant \|x\|$, $\alpha(x) \in (0, 1)$, and

$$\|\Phi(k+l, k)x\| \leqslant \gamma(x)\alpha(x)^l \quad \forall\, k,l \geqslant 0.$$

(P) For every $p > 0$, there exists a real functional $\sigma_p(\bullet) : \mathcal{X} \to \mathbb{R}$ such that

$$\sum_{l=0}^{\infty} \|\Phi(k+l, k)x\|^p \leqslant \sigma_p(x) \quad \forall\, k \geqslant 0 \quad \forall\, x \in \mathcal{X}.$$

(Q) For some $q > 0$, there exists a real functional $\sigma_q(\bullet) : \mathcal{X} \to \mathbb{R}$ such that

$$\sum_{l=0}^{\infty} \|\Phi(k+l, k)x\|^q \leqslant \sigma_q(x) \quad \forall\, k \geqslant 0 \quad \forall\, x \in \mathcal{X}.$$

(R) There exist real functionals $q(\bullet), \sigma(\bullet) : \mathcal{X} \to \mathbb{R}$ such that $q(x) > 0$ and

$$\sum_{l=0}^{\infty} \|\Phi(k+l, k)x\|^{q(x)} \leqslant \sigma(x) \quad \forall\, k \geqslant 0 \quad \forall\, x \in \mathcal{X}.$$

(S) For every $p > 0$, there exists a real functional $\mu_p(\bullet) : \mathcal{X} \to \mathbb{R}$ such that

$$\|\Phi(k+l, k)x\|^p \leqslant (l+1)^{-1}\mu_p(x) \quad \forall\, k,l \geqslant 0 \quad \forall\, x \in \mathcal{X}.$$

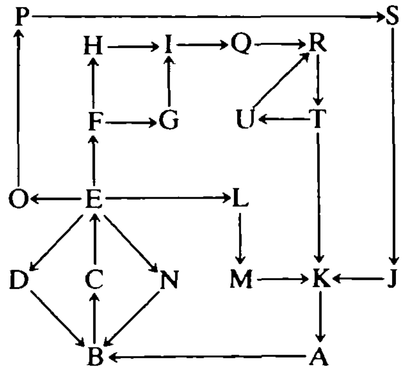
(T) For some $q > 0$, there exists a real functional $\mu_q(\bullet) : \mathcal{X} \to \mathbb{R}$ such that

$$\|\Phi(k+l, k)x\|^q \leqslant (l+1)^{-1}\mu_q(x) \quad \forall\, k,l \geqslant 0 \quad \forall\, x \in \mathcal{X}.$$

(U) There exist real functionals $q(\bullet), \mu(\bullet) : \mathcal{X} \to \mathbb{R}$ such that $q(x) > 0$ and

$$\|\Phi(k+l, k)x\|^{q(x)} \leqslant (l+1)^{-1}\mu(x) \quad \forall\, k,l \geqslant 0 \quad \forall\, x \in \mathcal{X}.$$

We shall verify the implications as indicated in the following diagram.



First we show that (C)$\Rightarrow$(E). Suppose (C) holds, and set

$$\alpha = \sup_{k\geq 0} \|\Phi(k+l_0, k)\|^{1/l_0} < 1.$$

It is readily verified by induction (on n) that

$$\sup_{k\geq 0} \|\Phi(k+nl_0, k)\| \leq \alpha^{nl_0} \quad \forall\, n \geq 0.$$

For each $l \geq 0$, let n_l be the least positive integer greater than or equal to l/l_0 (i.e. $0 \leq n_l - 1 \leq l/l_0 \leq n_l$), so that $0 \leq l-(n_l-1)l_0 \leq l_0$ and $\alpha^{n_l l_0} \leq \alpha^l$ for every $l \geq 0$. Thus, for any $k,l \geq 0$, we have

$$\begin{aligned}\|\Phi(k+l, k)\| &\leq \|\Phi(k+l, k+(n_l-1)l_0)\|\, \|\Phi(k+(n_l-1)l_0, k)\| \\ &\leq \sup_{k\geq 0} \|\Lambda(k)\|^{l-(n_l-1)l_0} \alpha^{(n_l-1)l_0} \\ &\leq \alpha^{-l_0} \max\left\{1, \sup_{k\geq 0} \|\Lambda(k)\|^{l_0}\right\} \alpha^l.\end{aligned}$$

Hence (C)$\Rightarrow$(E). Next we verify that (P)$\Rightarrow$(S). Let p be an arbitrary positive real number. For any $k,l \geq 0$, we have

$$\begin{aligned}(l+1)\, \|\Phi(k+l, k)x\|^p &= \sum_{m=0}^{l} \|\Phi(k+l, k)x\|^p \\ &\leq \sum_{m=0}^{l} \|\Phi(k+l, k+m)\|^p\, \|\Phi(k+m, k)x\|^p\end{aligned}$$

for all $x \in \mathcal{X}$. Note that (P) obviously implies that, for any $p > 0$,

$$\|\Phi(k+l, k)x\| \leq \sigma_p(x)^{1/p} < \infty$$

for all $x \in \mathcal{X}$ and every $k,l \geq 0$. Thus, by the Banach–Steinhaus theorem (e.g. see [2: p. 66]), there exists a positive constant η such that $\|\Phi(k+l, k)\| \leq \eta < \infty$ for every $k,l \geq 0$. Therefore, for any $k,l \geq 0$, we get

$$(l+1)\, \|\Phi(k+l, k)x\|^p \leq \eta^p \sigma_p(x) < \infty$$

for all $x \in \mathcal{X}$, whenever (P) holds. Hence (P)$\Rightarrow$(S). Proceeding in a similar fashion yields

$$(l+1)\|\Phi(k+l,k)\|^q \leqslant \sum_{m=0}^{l} \|\Phi(k+l,k+m)\|^q \, \|\Phi(k+m,k)\|^q \leqslant \rho_q^2 < \infty$$

for every $k,l \geqslant 0$, whenever (M) holds. Hence (M)$\Rightarrow$(K). By combining the two approaches used above, we finally supply a proof for (R)$\Rightarrow$(T). Suppose (R) holds and, for each $x \in \mathcal{X}$, let $n(x)$ be the least positive integer greater than or equal to $q(x)$ (i.e. $0 \leqslant n(x)-1 \leqslant q(x) \leqslant n(x) < \infty$), so that $0 \leqslant l \leqslant (l+1)^{n(x)}-1$ for all $x \in \mathcal{X}$ and every $l \geqslant 0$. Thus, for any $k,l \geqslant 0$, it follows that

$$\begin{aligned}(l+1)^{q(x)} \|\Phi(k+l,k)x\|^{q(x)} &\leqslant (l+1)^{n(x)} \|\Phi(k+l,k)x\|^{q(x)} \\ &= \sum_{m=0}^{(l+1)^{n(x)}-1} \|\Phi(k+l,k)x\|^{q(x)} \\ &\leqslant \sum_{m=0}^{(l+1)^{n(x)}-1} \|\Phi(k+l,k+m)\|^{q(x)} \, \|\Phi(k+m,k)x\|^{q(x)}\end{aligned}$$

for all $x \in \mathcal{X}$. Since (R) obviously implies that $\|\Phi(k+l,k)x\| \leqslant \sigma(x)^{1/q(x)} < \infty$ for all $x \in \mathcal{X}$ and every $k,l \geqslant 0$, we get $\|\Phi(k+l,k)\| \leqslant \eta < \infty$ (for some positive constant η) for every $k,l \geqslant 0$, by using the Banach–Steinhaus theorem again. Therefore, for any $k,l \geqslant 0$, we obtain

$$\begin{aligned}(l+1)\,\|\Phi(k+l,k)x\| &\leqslant \eta \Big(\sum_{m=0}^{\infty} \|\Phi(k+m,k)x\|^{q(x)} \Big)^{1/q(x)} \\ &\leqslant \eta\sigma(x)^{1/q(x)} < \infty\end{aligned}$$

for all $x \in \mathcal{X}$. Hence (R)$\Rightarrow$(T) with $q=1$. A straightforward application of the Banach–Steinhaus theorem yields (S)$\Rightarrow$(J) and (T)$\Rightarrow$(K). The remaining implications in Diagram 1 are trivial. Note that, since $\mathcal{B}[\mathcal{X}]$ is a Banach space, assertion (N) means (by definition) that the family of series $\{\sum_{l=0}^{\infty} \Phi(k+l,k) : k \geqslant 0\}$ is uniformly equiconvergent. □

3. Remarks

Remark 1. (Further equivalent assertions). The set of equivalent assertions in Theorem 1 is certainly not exhaustive. For instance, 'sup' (whenever it appears implicitly or explicitly in Theorem 1) can be replaced by 'limsup'. To illustrate this, we shall show that each of the assertions below, which will be required later in this paper, is also equivalent to the assertions (A–U)

(A′) $$\lim_{l\to\infty} \limsup_{k\to\infty} \|\Phi(k+l,k)\| = 0.$$

(D′) $$\limsup_{l\to\infty} \limsup_{k\to\infty} \|\Phi(k+l,k)\|^{1/l} < 1.$$

This is readily verified as follows. Consider the following auxiliary assertions: (B′) and (C′) obtained by changing '$\sup_{k\geqslant 0}$' to '$\limsup_{k\to\infty}$' in (B) and (C), respectively; and (E′) obtained by changing the requirement '$\forall\, k \geqslant 0$' to '$\forall\, k \geqslant k_0$

for some $k_0 \geq 0$' in (E). Note that (A)$\Rightarrow$(A')$\Rightarrow$(B')$\Rightarrow$(C') and (E')$\Rightarrow$(D')$\Rightarrow$(B') trivially. Moreover, it is a simple matter to show that (E')$\Rightarrow$(E). Finally, by applying exactly the same technique used to establish that (C)$\Rightarrow$(E) in the proof of Theorem 1, it can be shown that (C')$\Rightarrow$(E'). Therefore, each of (A') to (E') is equivalent to (A–U).

Remark 2. (Time-invariant systems). For the particular case of a constant operator sequence $(A(k) = A : k \in \mathbb{N})$, the associated equivalent assertions in Theorem 1 are trivially obtained by changing $\Phi(k+l, k)$ to A^l in the theorem statement. Many of these equivalent assertions for time-invariant systems are well known (e.g. see [6]). Among them, the following will be needed in the sequel (here $r(A)$ denotes the spectral radius of $A \in B[\mathcal{X}]$): For an arbitrary $A \in B[\mathcal{X}]$, the assertions below are equivalent.

($\bar{\text{A}}$) $\lim_{l\to\infty} \|A^l\| = 0$.

($\bar{\text{D}}$) $r(A) := \lim_{l\to\infty} \|A^l\|^{1/l} < 1$.

($\bar{\text{N}}$) $(\sum_{l=0}^{n} A^l : n \in \mathbb{N})$ converges uniformly.

($\bar{\text{N}}_0$) $\sum_{l=0}^{n} A^l \to (I - A)^{-1} \in B[\mathcal{X}]$ uniformly as $n \to \infty$.

Whereas ($\bar{\text{N}}$) is the time-invariant version of (N), the assertion ($\bar{\text{N}}_0$) has clearly no time-varying interpretation in general. However, it is well known (e.g. see [2: p. 567]) that ($\bar{\text{D}}$)$\Rightarrow$($\bar{\text{N}}_0$), and ($\bar{\text{N}}_0$)$\Rightarrow$($\bar{\text{N}}$) trivially. Note that the natural time-invariant counterpart of Definition 1 is: The autonomous version of model (1) (or equivalently an operator $A \in B[\mathcal{X}]$) is *uniformly asymptotically stable* if $\|A^l\| \to 0$ as $l \to \infty$.

Remark 3. (A brief review). The equivalence between (A) and (E) is well known (e.g. see [9]). Assertion (E) (resp. (O)) is usually referred to as *uniform* (resp. *strong*) *power equistability* (cf. [9–11]). The expression in the left hand side of (D) was called the generalized spectral radius of the sequence $(A(k) : k \in \mathbb{N})$ in [11], where the equivalence between (D) and (E) was analysed. Note that, according to Remarks 1 and 2, the inequality in (D') also generalizes the spectral radius condition in ($\bar{\text{D}}$). Assertion (I) (resp. (Q) and (R)) was referred to as *ℓ_q-uniform* (resp. *ℓ_q-strong* and *$\ell_q(x)$*) *equistability* in [11], in which was also established the equivalence between each of (E–G), (I), and (O–R), for the case of $p, q \geq 1$, by using the Baire category theorem. If we agree that "in this context a proof is 'elementary' if it does not use the Baire category theorem" (see [4: p. 13]) and recalling that (even in a Banach-space setting) the Banach–Steinhaus theorem has an 'elementary' proof (see [1: p. 98]), we conclude that the proofs presented here are substantially simpler than those in [11], since we have used (beyond really elementary analysis) just the Banach–Steinhaus theorem for establishing the equivalence between each of (A–U). Now consider the following assertion which, according to Definition 1, is naturally referred to as *strong asymptotic equistability*.

$$(\text{A}_s) \qquad \lim_{l\to\infty} \sup_{k\geq 0} \|\Phi(k+l, k)x\| = 0 \quad \forall x \in \mathcal{X}.$$

Note that (A)$\Rightarrow$(A$_s$) trivially, but (A$_s$)$\not\Rightarrow$(A) in general. Actually, (A$_s$)$\not\Rightarrow$(A) even for a constant sequence $(\Lambda(k)=\Lambda : k \in \mathbb{N})$, although (A$_s$)$\Rightarrow$(A) for the particular case of an operator sequence constantly equal to a compact operator (e.g. see [6]). However, (A$_s$)$\Rightarrow$(A) whenever $\sup_{k\geq 0} \|\Phi(k+l_0, k)\| < 1$ for some $l_0 \geq 1$ (recall that (C)$\Rightarrow$(A) in Theorem 1). By using the above result, it has been proved in [9] that (A$_s$)$\Rightarrow$(A) whenever $(\Lambda(k) : k \in \mathbb{N})$ is collectively compact (i.e. whenever the set

$$\bigcup_{k\geq 0} \{\Lambda(k)x : x \in \mathcal{X}, \|x\| \leq 1\}$$

is relatively compact in $\mathcal{X}$). Note that a constant sequence $(\Lambda(k)=\Lambda : k \in \mathbb{N})$, with $\Lambda \in \mathcal{B}[\mathcal{X}]$, is collectively compact if and only if Λ is compact.

Remark 4. (Time-varying convergent systems). If an operator sequence converges uniformly, then it shares the same stability properties with its limit. Precisely: Let $(\Phi(k+l, k) : k,l \in \mathbb{N})$ be the evolution operator process associated with an operator sequence $(\Lambda(k) : k \in \mathbb{N}) \in \mathcal{B}[\mathcal{X}]^{\mathbb{N}}$ which is supposed to converge uniformly to $\Lambda \in \mathcal{B}[\mathcal{X}]$. Since $\lim_{k\to\infty} \|\Lambda(k) - \Lambda\| = 0$, it is readily verified by induction on l that $\lim_{k\to\infty} \|\Phi(k+l, k) - \Lambda^l\| = 0$ for every $l \geq 0$. Hence

$$\lim_{k\to\infty} \|\Phi(k+l, k)\| = \|\Lambda^l\| \quad \forall\, l \geq 0.$$

By combining the above result with Remark 1 we get

$$(\mathrm{A}) \;\Leftrightarrow\; (\mathrm{A}') \;\Leftrightarrow\; (\bar{\mathrm{A}}) \;\Leftrightarrow\; \lim_{l\to\infty} \lim_{k\to\infty} \|\Phi(k+l, k)\| = 0,$$

with assertions (A), (A′), and ($\bar{\mathrm{A}}$) as in Theorem 1 and Remarks 1 and 2.

4. Forced systems

Given a sequence $\boldsymbol{u} = (u(i) : i \in \mathbb{N}) \in \mathcal{X}^{\mathbb{N}}$, consider another sequence $\boldsymbol{x} = (x(i) : i \in \mathbb{N})$ recursively defined by the following nonautonomous inhomogeneous difference equation.

$$x(i+1) = \Lambda(i)x(i) + u(i+1), \qquad x(0) = u(0), \tag{2a}$$

whose solution is

$$x(i) = \sum_{j=0}^{i} \Phi(i, j)u(j) \quad \forall\, i \geq 0, \tag{2b}$$

where $(\Phi(j+l, j) : j,l \in \mathbb{N})$ is the evolution operator process associated with the operator sequence $(\Lambda(i) : i \in \mathbb{N}) \in \mathcal{B}[\mathcal{X}]^{\mathbb{N}}$. The purpose of this section is to investigate some aspects of (uniform) input-state stability.

THEOREM 2 *Consider the model* (2). *The following assertions are equivalent.*

(a) $(\Lambda(i) : i \in \mathbb{N})$ *is uniformly asymptotically equistable.*

(b) *For every* $p \geq 1$, *there exists a positive number* λ_p *such that*

$$\|\boldsymbol{x}\|_p \leq \lambda_p \|\boldsymbol{u}\|_p \quad \forall\, \boldsymbol{u} \in \ell_p(\mathcal{X}).$$

(c) *For some* $q \geq 1$, *there exists a positive number* λ_q *such that*

$$\|\boldsymbol{x}\|_q \leq \lambda_q \|\boldsymbol{u}\|_q \quad \forall\, \boldsymbol{u} \in \ell_q(\mathcal{X}).$$

(d) *There exists a positve number* λ_∞ *such that*

$$\|\boldsymbol{x}\|_\infty \leq \lambda_\infty \|\boldsymbol{u}\|_\infty \quad \forall\, \boldsymbol{u} \in \ell_\infty(\mathcal{X}).$$

(e) $\boldsymbol{u} \in \ell_p(\mathcal{X}) \Rightarrow \boldsymbol{x} \in \ell_p(\mathcal{X})$ *for every* $p \geq 1$.

(f) $\boldsymbol{u} \in \ell_q(\mathcal{X}) \Rightarrow \boldsymbol{x} \in \ell_q(\mathcal{X})$ *for some* $q \geq 1$.

(g) $\boldsymbol{u} \in \ell_\infty(\mathcal{X}) \Rightarrow \boldsymbol{x} \in \ell_\infty(\mathcal{X})$.

(h) $\boldsymbol{u} \in \mathfrak{c}_0(\mathcal{X}) \Rightarrow \mathbf{x} \in \mathfrak{c}_0(\mathcal{X})$.

Moreover, the assertion below implies the above ones.

(i) $\boldsymbol{u} \in \mathfrak{c}(\mathcal{X}) \Rightarrow \mathbf{x} \in \mathfrak{c}(\mathcal{X})$.

Proof. If (a) holds true then, from (E) (cf. Theorem 1) and (2b) we get

$$\|x(i)\| \leq \gamma_i := \sum_{j=0}^{i} \alpha^{i-j}\beta_j \tag{3}$$

for some pair of constants $\gamma \geq 1$ and $\alpha \in (0, 1)$, with $\beta_i = \gamma\, \|u(i)\|$, for every $i \geq 0$. Recall that the convolution (or the Cauchy product) $c = (\gamma_i : i \in \mathbb{N}) = a \star b$ of a scalar sequence $a = (\alpha^i : i \in \mathbb{N})$ in ℓ_1 and a scalar sequence $b = (\beta_i : i \in \mathbb{N})$ in ℓ_p lies itself in ℓ_p, with $\|c\|_p \leq \|a\|_1\, \|b\|_p$, for any $p \geq 1$ (see [2; p. 529]); which clearly also holds if we set $p = \infty$. Hence, since $\|a\|_1 = \sum_{i=0}^{\infty} \alpha^i = (1-\alpha)^{-1}$, it follows from (3) that (b) and (d) hold, with $\lambda_p = \lambda_\infty = \gamma(1-\alpha)^{-1}$ for any $p \geq 1$. Thus (a) $\Rightarrow$ (b, d). Note that (b) $\Rightarrow$ (e) $\Rightarrow$ (f) and (d) $\Rightarrow$ (g) trivially. Now we show that (g) $\Rightarrow$ (d). For each $i \geq 0$, consider the transformation $\theta_i : \ell_\infty(\mathcal{X}) \to \mathcal{X}$ given, according to (2b), by

$$x(i) = \theta_i \boldsymbol{u} = \sum_{j=0}^{i} \Phi(i, j) u(j)$$

for all $\boldsymbol{u} = \big(u(j) : j \in \mathbb{N}\big) \in \ell_\infty(\mathcal{X})$, which is clearly linear and bounded (i.e. $\theta_i \in \mathcal{B}[\ell_\infty(\mathcal{X}), \mathcal{X}]$: actually $\|\theta_i\|_{\mathcal{B}[\ell_\infty(\mathcal{X}),\mathcal{X}]} \leq \sum_{j=0}^{i} \|\Phi(i, j)\|$). If (g) holds, then

$$\sup_{i \geq 0} \|\theta_i \boldsymbol{u}\| < \infty \quad \forall\, \boldsymbol{u} \in \ell_\infty(\mathcal{X}),$$

so that $\sup_{i\geq 0} \|\theta_i\|_{\mathcal{B}[\ell_\infty(\mathcal{X}),\mathcal{X}]} = \lambda_\infty$, for some positive constant λ_∞, by the Banach–Steinhaus theorem. Hence (d) holds, since $\sup_{i \geq 0} \|\theta_i \boldsymbol{u}\| \leq \lambda_\infty \|\boldsymbol{u}\|_\infty$ for all $\boldsymbol{u} \in \ell_\infty(\mathcal{X})$. Thus (g) $\Rightarrow$ (d). In a similar fashion we can show that (f) $\Rightarrow$ (c). For each $n \geq 0$ and an arbitrary $\boldsymbol{u} = \big(u(j) : j \in \mathbb{N}\big) \in \ell_q(\mathcal{X})$, set $\boldsymbol{x}_n = \big(x_n(i) : i \in \mathbb{N}\big)$, with

$$x_n(i) = \begin{cases} \sum_{j=0}^{i} \Phi(i, j) u(j) & \text{if } i \leq n, \\ 0 & \text{if } i > n, \end{cases}$$

so that $\boldsymbol{x}_n \in \ell_1(\mathcal{X}) \subseteq \ell_q(\mathcal{X})$; and consider the transformation $\Psi_n : \ell_q(\mathcal{X}) \to \ell_q(\mathcal{X})$ given by

$$\Psi_n \boldsymbol{u} = \boldsymbol{x}_n$$

for all $u \in \ell_q(\mathcal{X})$, which is clearly linear and bounded (i.e. $\Psi_n \in \mathbb{B}[\ell_q(\mathcal{X})]$: actually

$$\|\Psi_n\|_{\mathbb{B}[\ell_1(\mathcal{X})]} \leq \sum_{i=0}^{n} \max_{0 \leq j \leq i} \|\Phi(i, j)\|, \qquad \|\Psi_n\|^{q/(q-1)}_{\mathbb{B}[\ell_q(\mathcal{X})]} \leq \sum_{i=0}^{n} \sum_{j=0}^{i} \|\Phi(i, j)\|^{q/(q-1)},$$

for $q > 1$). If (f) holds, then

$$\text{(4)} \qquad \|x_n\|_q = \left(\sum_{i=0}^{n} \|x(i)\|^q\right)^{1/q} \leq \|x\|_q < \infty$$

for every $n \geq 0$, whenever $u \in \ell_q(\mathcal{X})$, according to (2b). Therefore,

$$\sup_{n \geq 0} \|\Psi_n u\|_q < \infty \quad \forall\, u \in \ell_q(\mathcal{X}),$$

so that $\sup_{n\geq 0} \|\Psi_n\|_{\mathbb{B}[\ell_q(\mathcal{X})]} = \lambda_q$, for some positive constant λ_q, by the Banach–Steinhaus theorem. Hence

$$\|x\|_q = \sup_{n \geq 0} \|\Psi_n u\|_q \leq \lambda_q \|u\|_q \quad \forall\, u \in \ell_q(\mathcal{X}),$$

according to (4). Thus (f) $\Rightarrow$ (c). On the other hand, each of (c) and (d) implies (a). To verify this, take an arbitrary $u \in \mathcal{X}$ and, for each $k \geq 0$, set

$$u_k = \big(u_k(i) : i \in \mathbb{N}\big) \in \ell_q(\mathcal{X}) \subset \ell_\infty(\mathcal{X})$$

for any $q \geq 1$ as follows:

$$u_k(i) = \begin{cases} u & \text{if } \ i = k, \\ 0 & \text{if } \ i \neq k, \end{cases}$$

so that $\|u_k\|_q = \|u_k\|_\infty = \|u\|$ for each $k \geq 0$ and any $q \geq 1$. Thus, for each $k \geq 0$, we get $x_k = \big(x_k(i) : i \in \mathbb{N}\big)$ from (2b), given by

$$x_k(i) = \begin{cases} \Phi(i, k)u & \text{if } \ i \geq k, \\ 0 & \text{if } \ 0 \leq i < k. \end{cases}$$

If (c) holds, then

$$\sum_{l=0}^{\infty} \|\Phi(k + l, k)u\|^q = \sum_{i=k}^{\infty} \|\Phi(i, k)u\|^q = \|x_k\|_q^q \leq \lambda_q^q \|u_k\|_q^q = \lambda_q^q \|u\|^q$$

for every $k \geq 0$ and all $u \in \mathcal{X}$, so that (I) (cf. Theorem 1) holds. Thus (c) $\Rightarrow$ (a). Now, for each $k \geq 0$, let $y_k = \big(y_k(i) : i \in \mathbb{N}\big)$ be recursively defined by model (2) with $u = \big(u(i) : i \in \mathbb{N}\big)$ replaced by $x_k = \big(x_k(i) : i \in \mathbb{N}\big)$, so that

$$y_k(i) = \begin{cases} (i - k + 1)\Phi(i, k)u & \text{if } \ i \geq k, \\ 0 & \text{if } \ 0 \leq i < k, \end{cases}$$

according to (2b). If (d) holds, then

$$\sup_{l \geq 0} (l + 1) \|\Phi(k + l, k)u\| = \sup_{i \geq k} (i - k + 1) \|\Phi(i, k)u\| = \|y_k\|_\infty$$
$$\leq \lambda_\infty \|x_k\|_\infty \leq \lambda_\infty^2 \|u_k\|_\infty = \lambda_\infty^2 \|u\|$$

for every $k \geq 0$ and all $u \in \mathcal{X}$, so that (K) (cf. Theorem 1) holds. Thus (d)$\Rightarrow$(a). Next we show that (a)$\Leftrightarrow$(h). First recall that

(5) $$\lim_{i\to\infty} \sum_{j=0}^{i} \alpha^{i-j}\xi_j = (1-\alpha)^{-1} \lim_{i\to\infty} \xi_i$$

for every convergent scalar sequence $(\xi_1 : i \in \mathbb{N})$ if and only if $|\alpha| < 1$ (e.g. see [7]). From (3) and (5) it follows that (a)$\Rightarrow$(h). On the other hand consider the following auxiliary assertions.

(h′) $$\boldsymbol{u} \in \mathfrak{c}_0(\mathcal{X}) \;\Rightarrow\; \boldsymbol{x} \in \ell_\infty(\mathcal{X}),$$

(h″) $$\|\boldsymbol{x}\|_\infty \leq \lambda_0 \|\boldsymbol{u}\|_\infty \quad \forall\, \boldsymbol{u} \in \mathfrak{c}_0(\mathcal{X}),$$

for some positive constant λ_0. We claim that

$$\text{(h)} \;\Rightarrow\; \text{(h′)} \;\Leftrightarrow\; \text{(h″)}.$$

Actually (h)$\Rightarrow$(h′)$\Leftarrow$(h″) trivially, and (h′)$\Rightarrow$(h″) as follows. For each $i \geq 0$, let $\theta_i^0 \in \mathbb{B}[\mathfrak{c}_0(\mathcal{X}), \mathcal{X}]$ be the restriction of $\theta_i \in \mathbb{B}[\ell_\infty(\mathcal{X}), \mathcal{X}]$ on the Banach space $\mathfrak{c}_0(\mathcal{X}) \subset \ell_\infty(\mathcal{X})$. Proceeding as in the proof of (g)$\Rightarrow$(d), we get

$$\sup_{i\geq 0} \|\theta_i^0\|_{\mathbb{B}[\mathfrak{c}_0(\mathcal{X}),\mathcal{X}]} = \lambda_0$$

for some positive constant λ_0, whenever (h′) holds, so that

$$\sup_{i\geq 0} \|\theta_i^0 \boldsymbol{u}\| \leq \lambda_0 \|\boldsymbol{u}\|_\infty$$

for all $\boldsymbol{u} \in \mathfrak{c}_0(\mathcal{X})$. Thus (h′)$\Rightarrow$(h″). Now, for an arbitrary $u \in \mathcal{X}$ and each $k \geq 0$, consider the sequences $\boldsymbol{u}_k$, $\boldsymbol{x}_k$, and $\boldsymbol{y}_k$ defined above. Note that

$$\|\boldsymbol{x}_k\|_\infty \leq \lambda_0 \|\boldsymbol{u}_k\|_\infty = \lambda_0 \|u\|$$

for every $k \geq 0$ whenever (h) holds, since (h)$\Rightarrow$(h″) and $\boldsymbol{u}_k \in \mathfrak{c}_0(\mathcal{X})$ for every $k \geq 0$; and

$$\|\boldsymbol{y}_k\|_\infty \leq \lambda_0 \|\boldsymbol{x}_k\|_\infty$$

for every $k \geq 0$ whenever (h) holds, since $\boldsymbol{x}_k \in \mathfrak{c}_0(\mathcal{X})$ for every $k \geq 0$ whenever (h) holds (because $\boldsymbol{u}_k \in \mathfrak{c}_0(\mathcal{X})$ for every $k \geq 0$) and (h)$\Rightarrow$(h″). Therefore,

$$\sup_{l\geq 0} (l+1) \|\Phi(k+l, k)u\| = \|\boldsymbol{y}_k\|_\infty \leq \lambda_0^2 \|u\|$$

for every $k \geq 0$ and all $u \in \mathcal{X}$, so that (K) (cf. Theorem 1) holds. Thus (h)$\Rightarrow$(a). Finally consider the following further auxiliary assertions.

(i′) $$\boldsymbol{u} \in \mathfrak{c}(\mathcal{X}) \;\Rightarrow\; \boldsymbol{x} \in \ell_\infty(\mathcal{X}),$$

(i″) $$\|\boldsymbol{x}\|_\infty \leq \lambda \|\boldsymbol{u}\|_\infty \quad \forall\, \boldsymbol{u} \in \mathfrak{c}(\mathcal{X})$$

for some positive constant λ; and let $\theta_i' \in \mathbb{B}[\mathfrak{c}(\mathcal{X}), \mathcal{X}]$ be the restriction of $\theta_i \in \mathbb{B}[\ell_\infty(\mathcal{X}), \mathcal{X}]$ on the Banach space $\mathfrak{c}(\mathcal{X}) \subset \ell_\infty(\mathcal{X})$, for each $i \geq 0$. Proceeding exactly as in the proof of (h)$\Rightarrow$(a), we get (i′)$\Leftrightarrow$(i″), so that (i)$\Rightarrow$(i″), which implies that (i)$\Rightarrow$(a). Note that (a)$\not\Rightarrow$(i); e.g. take $\Lambda(i)=0$ for even i and $\Lambda(i)=\frac{1}{2}I$ for odd i, and $u(i)=u\neq 0\in\mathcal{X}$ for all $i\geqslant 0$. □

COROLLARY 1. *Suppose* $(\Lambda(i): i\in\mathbb{N})\in\mathcal{B}[\mathcal{X}]^{\mathbb{N}}$ *converges uniformly to* $\Lambda\in\mathcal{B}[\mathcal{X}]$. *Then all the assertions* (a–i) *in Theorem* 2 *are equivalent. Moreover, they are also equivalent to the following further one*:

(j) $\exists\,(I-\Lambda)^{-1}\in\mathcal{B}[\mathcal{X}]$ and $\lim_{i\to\infty}x(i)=(I-\Lambda)^{-1}\lim_{i\to\infty}u(i)\quad\forall\,u\in\mathcal{C}(\mathcal{X})$.

Proof. According to Theorem 2, we just need to show that (a)$\Rightarrow$(j), since (j)$\Rightarrow$(i) trivially. First recall that (a)$\Rightarrow$(h). Now take the limiting operator $\Lambda\in\mathcal{B}[\mathcal{X}]$ and, for an arbitrary $u\in\mathcal{X}$, set

$$\tilde{x}(i)=\sum_{j=0}^{i}[\Phi(i,j)-\Lambda^{i-j}]u,\qquad \tilde{u}(i+1)=[\Lambda(i)-\Lambda]\sum_{j=0}^{i}\Phi(i,j)u,$$

for each $i\geqslant 0$. It is readily verified that

$$\tilde{x}(i+1)=\Lambda\tilde{x}(i)+\tilde{u}(i+1)\quad\forall\,i\geqslant 0,\qquad \tilde{x}(0)=\tilde{u}(0)=0.$$

If (a) holds and $\|\Lambda(i)-\Lambda\|\to 0$ as $i\to\infty$, then the above autonomous model is uniformly asymptotically stable, according to Remark 4, so that (h) also applies to it in particular. Moreover, $\lim_{i\to\infty}\|\tilde{u}(i)\|=0$ whenever $\lim_{i\to\infty}\|\Lambda(i)-\Lambda\|=0$ and (a) holds (since (a)$\Rightarrow$(L) in Theorem 1). Thus

$$\lim_{i\to\infty}\left\|\sum_{j=0}^{i}[\Phi(i,j)-\Lambda^{i-j}]u\right\|=\lim_{i\to\infty}\|\tilde{x}(i)\|=0. \tag{6}$$

On the other hand, (2b) gives

$$x(i)-\sum_{j=0}^{i}\Lambda^{i-j}u=\sum_{j=0}^{i}\Phi(i,j)[u(j)-u]+\sum_{j=0}^{i}[\Phi(i,j)-\Lambda^{i-j}]u,$$

for any $u\in\mathcal{X}$ and every $i\geqslant 0$, so that, if (a) holds, then (E) gives (cf. Theorem 1)

$$\left\|x(i)-\sum_{j=0}^{i}\Lambda^{i-j}u\right\|\leqslant\sum_{j=0}^{i}\alpha^{i-j}\zeta_j+\left\|\sum_{j=0}^{i}[\Phi(i,j)-\Lambda^{i-j}]u\right\|,$$

where $\zeta_i=\gamma\,\|u(i)-u\|$, for every $i\geqslant 0$, with $\gamma\geqslant 1$ and $\alpha\in(0,1)$ as in (3). Hence, (5) and (6) give

$$\lim_{i\to\infty}\|u(i)-u\|=0\ \Rightarrow\ \lim_{i\to\infty}\left\|x(i)-\sum_{j=0}^{i}\Lambda^{j}u\right\|=0$$

whenever (a) holds. Moreover, since $\Lambda(i)\to\Lambda\in\mathcal{B}[\mathcal{X}]$ uniformly as $i\to\infty$, it follows from Remarks 2 and 4 that (a) implies ($\bar{\mathrm{N}}_0$) in Remark 2. Thus (a)$\Rightarrow$(j), since

$$\|x(i)-(I-\Lambda)^{-1}u\|\leqslant\left\|x(i)-\sum_{j=0}^{i}\Lambda^{j}u\right\|+\left\|\sum_{j=0}^{i}\Lambda^{j}-(I-\Lambda)^{-1}\right\|\,\|u\|$$

for any $u\in\mathcal{X}$ and every $i\geqslant 0$. □

5. Concluding Remarks

Remark 5. (ℓ_p-input ℓ_∞-state stability (with $p>1$), $\mathfrak{c}_0$-input ℓ_∞-state stability, and $\mathfrak{c}$-input ℓ_∞-state stability). For $q>1$, the result (f)$\Rightarrow$(a) in Theorem 2 becomes a particular case of that presented in [10]. Indeed, each of the assertions below is also equivalent to (a–h).

(b′) For every $p>1$ there exists a positive number η_p such that

$$\|x\|_\infty \leq \eta_p \|u\|_p \quad \forall\, u \in \ell_p(\mathcal{X}).$$

(c′) For some $q>1$ there exists a positive number η_q such that

$$\|x\|_\infty \leq \eta_q \|u\|_q \quad \forall\, u \in \ell_q(\mathcal{X}).$$

(e′) $u \in \ell_p(\mathcal{X}) \Rightarrow x \in \ell_\infty(\mathcal{X})$ for every $p>1$.

(f′) $u \in \ell_q(\mathcal{X}) \Rightarrow x \in \ell_\infty(\mathcal{X})$ for some $q>1$.

(h′) $u \in \mathfrak{c}_0(\mathcal{X}) \Rightarrow x \in \ell_\infty(\mathcal{X})$.

(h″) $\|x\|_\infty \leq \eta_0 \|u\|_\infty$ for all $u \in \mathfrak{c}_0(\mathcal{X})$, for some positive constant η_0.

(i′) $u \in \mathfrak{c}(\mathcal{X}) \Rightarrow x \in \ell_\infty(\mathcal{X})$.

(i″) $\|x\|_\infty \leq \eta \|u\|_\infty$ for all $u \in \mathfrak{c}(\mathcal{X})$, for some positive constant η.

This can be verified as follows. From (E) (see Theorem 1) and (2b), we get (a)$\Rightarrow$(b′) by using the Hölder inequality. Note that (b′)$\Rightarrow$(e′)$\Rightarrow$(f′) trivially. For each $i \geq 0$, let the transformation $\theta_{i,q} : \ell_q(\mathcal{X}) \to \mathcal{X}$ be the restriction of $\theta_i : \ell_\infty(\mathcal{X}) \to \mathcal{X}$ on $\ell_q(\mathcal{X})$, which is clearly linear and bounded (i.e. $\theta_{i,q} \in \mathfrak{B}[\ell_q(\mathcal{X}), \mathcal{X}]$: actually

$$\|\theta_{i,q}\|^{q/(q-1)}_{\mathfrak{B}[\ell_q(\mathcal{X}),\mathcal{X}]} \leq \sum_{j=0}^{i} \|\Phi(i,j)\|^{q/(q-1)}$$

for any $q>1$). Proceeding as in the proof of (g)$\Rightarrow$(d) in Theorem 2, we get (f′)$\Rightarrow$(c′). It can be shown that (c′)$\Rightarrow$(a) by using the same technique proposed in [10], which is essentially the following. Let $\varepsilon \in (0,1)$ and, for an arbitrary $u \in \mathcal{X}$ and for each $k \geq 0$, set

$$u'_k(i) = \begin{cases} \varepsilon^{i-k}\Phi(i,k)u & \text{if } i \geq k, \\ 0 & \text{if } 0 \leq i < k. \end{cases}$$

For each $k \geq 0$, let $x'_k = (x'_k(i) : i \in \mathbb{N})$ be the response to $u'_k = (u'_k(i) : i \in \mathbb{N})$ in (2b), so that

$$x'_k(k+l) = (1-\varepsilon)^{-1}(1-\varepsilon^{l+1})\Phi(k+l,k)u \quad \forall\, k,l \geq 0.$$

Note that (c′) implies $\|\Phi(k+l,k)\| \leq \eta_q$ for every $k,l \geq 0$ (consider the sequences u_k and x_k defined in the proof of Theorem 2), which implies

$$\|u'_k\|_q \leq \eta_q(1-\varepsilon^q)^{-1/q}\|u\|.$$

Hence $\|x'_k(k+l)\| \leq \|x'_k\|_\infty \leq \eta_q^2(1-\varepsilon^q)^{-1/q}\|u\|$ whenever (c′) holds. Therefore

$$\sup_{k\geq 0} \|\Phi(k+l,k)\| \leq \eta_q^2(1-\varepsilon)(1-\varepsilon^q)^{-1/q}(1-\varepsilon^{l+1})^{-1} \quad \forall\, l \geq 0.$$

Since $q>1$, we have $(1-\varepsilon)(1-\varepsilon^q)^{-1/q}\to 0$ as $\varepsilon\to 1$. Then, by taking $\varepsilon=\varepsilon_1\in(0,1)$ sufficiently close to 1, there exists $l_1=l_1(\varepsilon_1)$ large enough so that (B) (cf. Theorem 1) holds true. Thus (c′)⇒(a). Finally note that (g)⇒(i′)⇒(h′)⇒(e′) and (d)⇒(i″)⇒(h″)⇒(h′) trivially.

Remark 6. (ℓ_1-input ℓ_∞-state stability). By setting $\Lambda(k)=I\in\mathcal{B}[\mathcal{X}]$ for every $k\geqslant 0$, it is trivially verified that (f′)$\not\Rightarrow$(a) for $q=1$ in Remark 5. Actually, the assertions below are equivalent.

(a_1) $\|\Phi(k+l,k)\|\leqslant\eta$ for all $k,l\geqslant 0$, for some positive constant η.

(c_1) $\|x\|_\infty\leqslant\eta\,\|u\|_1$ for all $u\in\ell_1(\mathcal{X})$, for some positive constant η.

(f_1) $u\in\ell_1(\mathcal{X})\Rightarrow x\in\ell_\infty(\mathcal{X})$.

The verification is straightforward: (a_1)⇒(c_1)⇒(f_1) trivially; (c_1)⇒(a_1) by using the sequences u_k and x_k defined in the proof of Theorem 2; and (f_1)⇒(c_1) as follows. Let $\theta_{i,1}\in\mathcal{B}[\ell_1(\mathcal{X}),\mathcal{X}]$ be defined as in Remark 5 (so that $\|\theta_{i,1}\|_{\mathcal{B}[\ell_1(\mathcal{X}),\mathcal{X}]}\leqslant\max_{0\leqslant j\leqslant i}\|\Phi(i,j)\|$), and proceed as in the proof of (g)⇒(d) in Theorem 2.

Remark 7. (An illustrative application). Suppose a given operator sequence $(\Lambda(i):i\in\mathbb{N})\in\mathcal{B}[\mathcal{X}]^{\mathbb{N}}$ converges uniformly to $\Lambda\in\mathcal{B}[\mathcal{X}]$, and consider the state sequences $x=(x(i):i\in\mathbb{N})\in\mathcal{X}^{\mathbb{N}}$ and $\bar{x}=(\bar{x}(i):i\in\mathbb{N})\in\mathcal{X}^{\mathbb{N}}$ generated by the models

(2a) $$x(i+1)=\Lambda(i)x(i)+u(i+1),\qquad x(0)=u(0),$$

(2ā) $$\bar{x}(i+1)=\Lambda\bar{x}(i)+\bar{u}(i+1),\qquad \bar{x}(0)=\bar{u}(0),$$

for arbitrary bounded input sequences $u=(u(i):i\in\mathbb{N})\in\ell_\infty(\mathcal{X})$ and $\bar{u}=(\bar{u}(i):i\in\mathbb{N})\in\ell_\infty(\mathcal{X})$, respectively. We claim that the nonautonomous uniformly convergent model (2a) is an asymptotic state estimator for the limiting autonomous model (2ā) (i.e. $\lim_{i\to\infty}\|x(i)-\bar{x}(i)\|=0$ whenever $\lim_{i\to\infty}\|u(i)-\bar{u}(i)\|=0$) if and only if the limiting autonomous model (2ā) is uniformly asymptotically stable. In other words, by setting $\tilde{u}=u-\bar{u}$ and $\tilde{x}=x-\bar{x}$, we claim that the assertions below are equivalent.

(ā) $\Lambda\in\mathcal{B}[\mathcal{X}]$ is uniformly asymptotically stable (cf. Remark 2).

($\bar{h}$) $\tilde{u}\in c_0(\mathcal{X})\Rightarrow\tilde{x}\in c_0(\mathcal{X})$.

This can be verified as follows. First recall that (ā) is equivalent to (a) in Theorem 2, according to Remark 4, since $\lim_{i\to\infty}\|\Lambda(i)-\Lambda\|=0$. By setting $\tilde{\Lambda}(i)=\Lambda(i)-\Lambda$ for each $i\geqslant 0$, we get from (2a) and (2ā) the model

(2ã) $$\tilde{x}(i+1)=\Lambda\tilde{x}(i)+w(i+1),\qquad \tilde{x}(0)=\tilde{u}(0)=w(0),$$

with $w=(w(i):i\in\mathbb{N})\in\mathcal{X}^{\mathbb{N}}$ given by $w(i+1)=\tilde{\Lambda}(i)x(i)+\tilde{u}(i+1)$ for each $i\geqslant 0$. Now suppose (ā) holds. Recall that (a) implies (g) in Theorem 2, so that $x\in\ell_\infty(\mathcal{X})$, since $u\in\ell_\infty(\mathcal{X})$. Hence, $\tilde{u}\in c_0(\mathcal{X})$ implies $w\in c_0(\mathcal{X})$, since $\lim_{i\to\infty}\|\tilde{\Lambda}(i)\|=0$, which implies $\tilde{x}\in c_0(\mathcal{X})$ in (2ã) whenever (ā) holds, according to Theorem 2 (for the particular case of a constant sequence). Thus (ā)⇒($\bar{h}$). On the other hand, set $\bar{u} = 0$ in (2ā), so that $\tilde{u} = u$ and $\tilde{x} = x$. Hence ($\bar{\text{h}}$) implies that $x \in \mathfrak{c}_0(\mathcal{X})$ whenever $u \in \mathfrak{c}_0(\mathcal{X})$; or equivalently, ($\bar{\text{h}}$) implies (h) in Theorem 2, which implies (a) according to Theorem 2. Thus ($\bar{\text{h}}$) $\Rightarrow$ (ā).

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